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On a Randomized Romberg Integration Method

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Abstract. *We demonstrate how symmetrized Riemann integration recently introduced in [3] can be employed in the evaluation of integrals of piecewise continuous differentiable functions. A Romberg symmetrized integration method, with randomized intervals, is developed for various possible applications. Integration error bounds are derived statistically for the pertaining Richardson extrapolations. Symmetrized-to-conventional integral ratio computations are suggested as a means to reduce these error bounds.*

Key words: Romberg integration, Randomized intervals, Error bounds, Symmetrized integral, Functional equations.

AMS Subject Classifications : 65Q05, 65G40, 65C50

1. Introduction

This work deals with the problem of integrating over $I = [a,b] \subset R = (-\infty,\infty)$ the piecewise continuous real function

$$f(x) = \sum_{l=1}^{L} f_l(x)\kappa_l(x) \quad with \quad \kappa_l(x) = \begin{cases} 1, \ x \in I_l = [a_l, b_l^-] \\ 0, \ x \notin I_l \end{cases}, \tag{1}$$

b_l^- as the left-hand-side limit of b_l, and $f_l(x)$ is, $\forall\ l$, either a monotonically increasing or a monotonically decreasing differentiable function over I_l. Clearly, the interval set $\{I_l\}_{l=1}^{L}$ is here a partition of I. It reports on a certain Romberg method of integration that utilizes the concept of the symmetrized integration [3], over I_l , of such $f_l(x)$ functions. Due to the novely of this concept, the rest of this section shall be a brief survey of it.

The Riemann integral of a function $y = f_l(x)$over an interval I_l is

$$\rho_l = \int_{I_l} f_l(x) \quad dx, \tag{2}$$

is based on the projection of the curve of $f_l(x)$ on the $x-$ axis $(y = 0)$ of the Cartesian plane. Let us assume that the range of $f_l(x)$, that corresponds to $[a_l, b_l^-]$, is $[f_l(a_l), f_l(b_l^-)] = [c_l, d_l^-] \subset R$ in order to state the definitions that follow.

Definition 1.1. The symmetrized integral [3] of the $y = f_l(x)$ function is

$$\gamma_l = \int_{J_l} \eta_l(\tau) \quad d\tau, \tag{3}$$

with $\eta_l(\tau) = z$ obtained from $f_l(x) = y$ by means of a linear transformation $\Gamma : R^2 \to R^2$, such that $\Gamma(x,y)^T = \mathfrak{C}(x,y)^T = (\tau,z)^T$, with $\mathfrak{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, a rotation matrix, and $J_l = [\alpha_l, \beta_l]$ satisfying the affine map

$$(\alpha_l, \beta_l)^T = \mathfrak{C}(a_l, b_l^-)^T + \frac{1}{\sqrt{2}}(c_l - b_l^-, d_l^- + a_l)^T. \tag{4}$$

Clearly γ_l is another Riemann integral of $f_l(x)$ that is based on the projection of the curve of $f_l(x)$, over the $[\alpha_l, \beta_l]$ interval, on the median $(y = x)$ of the first quadrant of the Cartesian plane. This takes place instead of projection of $f_l(x)$ on the $y = 0$ axis in the $\rho_l -$ integral. The symmetrized $\gamma_l -$ integral should, nontheless, by no means be confused with a changed variable form [7], for the same ρ_l of $f_l(x)$.

Lemma 1.1. [3] *If $f_l(x)$ is monotonically increasing or decreasing differentiable function over $[a_l, b_l^-]$ then $\eta_l(\tau)$ is a one-to one map over $[\alpha_l, \beta_l]$.*

Definition 1.2. Let $f_l(x)$ be a monotonically increasing or decreasing function over $[a_l, b_l^-]$. Then an end point variation [3] of $f_l(x)$ at $x = a_l$ is

$$\varsigma_{a_l} = \varsigma[a_l, f(a_l)] = \tfrac{1}{4}(a_l^2 + a_l c_l - c_l^2) = \tfrac{1}{4}[a_l - c_l(\sqrt{2} - 1)][a_l + c_l(\sqrt{2} + 1)]. \tag{5}$$

Based on the fundamental theorem that follows, the previous ς_{a_l} number turns out to play the role of a generalized integrated "antiderivative" in the calculus of the $\rho_l - \gamma_l$ difference integral.

Theorem 1.1. [3] *(Differenced integral calculus) If ρ_l is the conventional integral of a monotonically increasing or decreasing function $f_l(x)$ over $[a_l, b_l^-]$ and γ_l is its pertaining symmetrized integral, then*

$$\gamma_l - \rho_l = \varsigma_{b_l^-} - \varsigma_{a_l}. \tag{6}$$

This result is similar to, though different from, the Fundamental Theorem of Calculus. It indicates remarkably that one needs to know or compute only one of the two integrals ρ_l or γ_l (whichever is easier) then use this theorem to find the other one. Nevertheless, the $\gamma_l -$ integral can serve either as an alternative or a complement to the $\rho_l -$ integral in some graphical or charting applications. Indeed, the symmetrized-to-conventional integral ratio γ_l/ρ_l could have some specific technical applications. Indeed if

(i) $\rho_l > 0$ and $\gamma_l > 0$, then the graph of $f_l(x)$ is in the sector, of the first quadrant, which is above the median,

(ii) $\rho_l > 0$ and $\gamma_l < 0$, then the graph of $f_l(x)$ is in the sector, of the first quadrant, which is below the median,

(iii) $\rho_l < 0$ then $\gamma_l < \rho_l < 0$, the graph of $f_l(x)$ is in the fourth quadrant.
Therefore the situations (i) and (iii) are characterized by a $\gamma_l/\rho_l > 0$ while the situation (ii) corresponds to $\gamma_l/\rho_l < 0$. Moreover, computations of the γ_l/ρ_l ratios can in principle have some other useful applications in various fields. One of these applications, which will be demonstrated later in this paper, shall aim at reducing the error in evaluating ρ_l by the Romberg method of numerical integration.

The rest of the paper is organized as follows. Motivated by interest in symmetrized integral, section 2 shall illustrate how the Romberg method, based on the trapezoid rule, turns out to be ideal for the computation of this integral. Section 3 is a study on randomization of intervals in Romberg integration as a way to elevate correlation coefficients for the Richardson extrapolation lines with scatterplots of relatively small size. It reports, moreover, on statistical error bounds for this novel method of integration. A possible enhancement of the computation of the ρ_l – integral, which is based on utilization of Theorem 1.1 is provided in section 4 ; to be followed in section 5 by some conclusions.

2. Evaluation of $\eta_l(\tau)$

In most activities involving a Riemann integration, $f_l(x)$ would be a given analytical or discrete function. In contrast, $\eta\ (\tau)$, in the form

$$\Pi_l(\tau, z) = z - \eta_l(\tau) = 0, \tag{7}$$

needs to be derived from

$$H_l(x, y) = y - f_l(x) = 0, \tag{8}$$

by means of the Γ –map. This happens to lead to the result that follows.

Theorem 2.1. [4] *The integrand $z = \eta_l(\tau)$ of the symmetrized integral (2) of a monotonically increasing or decreasing differentiable $f_l(x)$ over $[a_l, b_l^-]$ is a solution to the functional equation*

$$f_l\left[\tfrac{1}{\sqrt{2}}(\tau - z)\right] + f_l^{-1}\left[\tfrac{1}{\sqrt{2}}(\tau + z)\right] = \sqrt{2}\,\tau. \tag{9}$$

Example 2.1. For the straight line function
$y = f_l\ (x) = e_l\, x + q_l,\ |\ e_l\ | \neq 1,$ the corresponding functional equation (9) is
$e_l\ \tfrac{1}{\sqrt{2}}(\tau - z) + q_l + \dfrac{1}{e_l}\ \tfrac{1}{\sqrt{2}}(\tau + z) - \dfrac{q_l}{e_l} = \sqrt{2}\,\tau,$ and this can easily be rearranged to

$$z = \eta_l(\tau) = \frac{e_l - 1}{e_l + 1}\,\tau + \frac{\sqrt{2}}{e_l + 1}\,q_l. \tag{10}$$

Example 2.2. Consider next the simplest nonlinear function, i.e. the monomial : $y = f_l\ (x) = x^k,\ x \geq 0,$ with $k >> 1$. Its corresponding functional equation,
$\left[\tfrac{1}{\sqrt{2}}(\tau - z)\right]^k + \left[\tfrac{1}{\sqrt{2}}(\tau + z)\right]^{1/k} = \sqrt{2}\,\tau,$ cannot be resolved analytically in z, as an explicit function of τ ; contrary to the situation in Example 1.

Unfortunately the same nonresolvable situation happens to hold as well for all other nonlinear functions like, e.g., the exponential $f_l\ (x) = k^x$ for which $k^{\frac{1}{\sqrt{2}}(\tau - z)}$ + $\log_k \tfrac{1}{\sqrt{2}}(\tau + z) = \sqrt{2}\,\tau.$

2.1. **Broken line approximation to** $\eta_l(\tau)$

In view of the previous examples and related facts, any analogue or digital nonlinear $y = f_l(x)$ can be approximated to a system of broken lines vis

$$f_l(x) \approx \sum_{i=1}^{n_l} f_{li}(x)\vartheta_{li}(x), \tag{11}$$

where $f_{li}(x) = e_{li}\,x + h_{li}$, $\vartheta_{li}(x) = \begin{cases} 1, & x \in I_{li} = [a_{li}, b_{li}^-] \\ 0, & x \notin I_{li} \end{cases}$, $f_l(a_{li}) = c_{li}$ and $f_l(b_{li}^-) = d_{li}^-$.

Making use of (10) allows directly for the corresponding broken-line approximate representation

$$\eta_l(\tau) \approx \sum_{i=1}^{n_l} \eta_{li}(\tau)\chi_{li}(\tau), \tag{12}$$

in which $\eta_{li}(\tau) = \dfrac{e_{li}-1}{e_{li}+1}\,\tau + \dfrac{\sqrt{2}}{e_{li}+1}\,q_{li}$, $\chi_{li}(\tau) = \begin{cases} 1, & \tau \in J_{li} = [\alpha_{li}, \beta_{li}] \\ 0, & \tau \notin J_{li} \end{cases}$, and $(\alpha_{li}, \beta_{li})^T = \mathfrak{C}\,(a_{li}, b_{li}^-)^T + \frac{1}{\sqrt{2}}(c_{li} - b_{li}^-, d_{li}^- + a_{li})^T$.

2.2. **Approximate symmetrized-to-conventional integral ratio**

The broken line representation (12) of $\eta_l(\tau)$defines a certain sectioning scheme with an effective step size of $h_l = \dfrac{b_l^- - a_l}{n_l}$, even when the sectioning interval I_{li} is not uniform or random. Obviously, utilization of (12) in (2) provides, when $\upsilon_l = \dfrac{1}{n_l}$, for an estimate for γ_l, as an alternative way for estimation of ρ_l via (9), which is

$$\gamma_{l\,n_l} = \gamma_l(\upsilon_l) = \sum_{i=1}^{n_l} \int_{\alpha_{li}}^{\beta_{li}} \left[\frac{e_{li}-1}{e_{li}+1}\,\tau + \frac{\sqrt{2}}{e_{li}+1}\,q_{li} \right] \chi_{li}(\tau)\,d\tau,$$

or

$$\gamma_l(\upsilon_l) = \sum_{i=1}^{n_l} \frac{1}{2}\,\frac{e_{li}-1}{e_{li}+1}\,(\beta_{li}^2 - \alpha_{li}^2) + \frac{\sqrt{2}}{e_{li}+1}\,q_{li}\,(\beta_{li} - \alpha_{li}).$$

A finer sectioning scheme involving the use of larger n_l and smaller I_{li} intervals generates the finer $\gamma_l(\upsilon_l)$ estimate which is expected to tend, in principle, to γ_l when $n_l \to \infty$, i.e., when $\upsilon_l \to 0$.

A practical approach to this objective consists of course in a standard application of the equivalent Romberg integration [2] with a linear (or nonlinear) Richardson's extrapolation [5] on an $\upsilon_l - \gamma_l(\upsilon_l)$ plane, which yields $\gamma_l(0) = \gamma_l$.

In a similar fashion, it is possible to compute the symmetrized-to-conventional integral ratio

$$\frac{\gamma_l}{\rho_l}(\upsilon_l) = \frac{\sum_{i=1}^{n_l} \frac{1}{2}\frac{e_{li}-1}{e_{li}+1}(\beta_{li}^2-\alpha_{li}^2) + \frac{\sqrt{2}}{e_{li}+1}q_{li}(\beta_{li}-\alpha_{li})}{\sum_{i=1}^{n_l} \frac{1}{2}e_{li}\,(b_{li}^{-2}-a_{li}^2) + q_{li}\,(b_{li}^{-}-a_{li})}$$

which is supposed to yield $\frac{\gamma_l}{\rho_l} = \frac{\gamma_l}{\rho_l}(0)$. Furthermore, it is anticipated that the value of $\frac{\gamma_l}{\rho_l}(0)$ is less sensitive to the nature of the employed Richardson extrapolation scheme than the value of $\gamma_l(0)$ to enable utilization of the procedure that follows.

Proposition 2.1. *If the error of computing* $\frac{\gamma_l}{\rho_l}(0) <$ *the error of computing* $\rho_l(0)$, *then the estimated accuracy of* $\rho_l(0)$ *is enhanced when*

$$\rho(0) = (\varsigma_{b_l^-} - \varsigma_{a_l})/[\frac{\gamma_l}{\rho_l}(0) - 1], \tag{13}$$

and $\frac{\gamma_l}{\rho_l}(0)$ *is the only quantity to be computed.*

Proof. This is in fact a corollary of Theorem 1.1, combined with realizing that if $\rho_l(0) \pm \lambda$ and $\gamma(0) \pm \lambda$, in the right hand side of (13), have the same error λ, then the corresponding ratio result is $\frac{\gamma_l}{\rho_l}(0) \pm \frac{\lambda}{\rho_l(0)}$. This would be the effective error of $\rho_l(0)$ in the left hand side of (13), i.e., ending up with an error reduction by a factor of $\frac{1}{\rho_l(0)}$. ■

3. A randomized Romberg integration method over I_l

Employment of Romberg integration, based on a composite trapezoid rule, for the evaluation of the $\rho_l(\upsilon_l)$ and/or the $\frac{\gamma_l}{\rho_l}(\upsilon_l)$ ratios over I_l implements a construction of a Romberg table, which is known [2] to be designed to eliminate the systematic error gradually. Despite the high accuracy of the Romberg method, it suffers from a serious problem of slow convergence which calls for increased values of n_l, number of discrete points.

To circumvent this problem we shall develop here a small size Romberg integration scheme (over I_l where the integral ρ_l of $f_l(x)$ is symmetrizable to γ_l) which employs randomization of the size of intervals for a finite set of values for n_l as an alternative to increasing the size of n_l .

Consider then subdivision of the $I_l = [a_l, b_l^-]$ interval according to

$$x_{i-1,k,r} + h_{l,i-1,k,r} = x_{i,k,r} \tag{14}$$

where $i = 1,2,3,\ldots\ldots,n_l$ while $k = k_0, k_0+1, k_0+2, k_0+3,\ldots\ldots,k_0+N_l-1$ stands for the number of subdivision intervals $k = 1 + \ln n_l/\ln 2$ of $n_l = 2^{k-1}$ and $r = 1,2,3,\ldots\ldots,M_l$ stands for a number enumerating a stochastic set of arbitrary discrete random probabilities

$\wp_l = \{p_1, p_2, p_3, \ldots\ldots, p_r, \ldots, p_{M_l}\} \subset (0,1)$. For consistency it is anticipated that k_0 is a small number between 5 and 10, say, and assuming that $m = k - k_0 + 1$ provides a range of $[1, N_l]$ for the number of extrapolation points where for reasons of computational economy, N_l is intended to be a sufficiently small number. The elements of $\wp_l$, generated from a set of appropriate random numbers, define Binomial [1] probability density distributions

$$\theta_{l,j,k,r} = \binom{n_l - 1}{j} p_r^j \,(1 - p_r)^{n_l - j - 1} \quad , j = i - 1 = 0, 1, 2, 3, \ldots\ldots, n_l - 1 \tag{15}$$

for the intervals

$$h_{l,j,k,r} = \theta_{l,j,k,r}(b_l^- - a_l), \tag{16}$$

in a randomized interval subdivision scheme (13) for which

$$I_{l,j,k,r} = [a_{l,j,k,r}, b_{l,j,k,r}] \in [a_l, b_l^-], \text{ with } h_{l,j,k,r} = b_{l,j,k,r} - a_{l,j,k,r} = x_{j+1,k,r} - x_{j,k,r} \text{ and}$$
$$x_{0,k,r} = a_l \quad \text{and} \quad x_{n_l,k,r} = b_l^-, \forall\, k \text{ and } r. \tag{17}$$

In this procedure j coincides, for any fixed r, with the number of successes out of $n_l - 1$ trials in a Bernoulli (Binomial) experiment with a probability of success equalling to p_r.

Romberg integration of $f_l(x)$ over I_l, based on the composite trapezoid rule [2], allows in the present context, for writing

$$\rho_{l,k,r} = \tfrac{1}{2}\left[\; h_{l,0,k,r}\; f(a_l) \;+\; h_{l,n_l-1,k,r}\; f(b_l^-) + \sum_{i=1}^{n_l-1}(h_{l,i-1,k,r} + h_{l,i,k,r})\; f\Big[a_l + \sum_{j=0}^{i-1} h_{l,j,k,r}\Big] \right] + \varkappa$$

with n_l in the summation limit representing 2^{k-1}, and $\varkappa$ being the error term

$$\varkappa = O\Big[\,(h^2_{l,0,k,r}\,,\; h^2_{l,n_l-1,k,r})\,\Big] = -\tfrac{1}{12}\Big[\,h^2_{l,0,k,r} f_l'(a_l) \;-\; h^2_{l,n_l-1,k,r} f_l'(b_l^-)\,\Big] + \text{terms of higher orders}$$

of $h_{l,0,k,r}$ and $h_{l,n_l-1,k,r}$. It should be noted here that in many situations the $f_l'(a_l)$ and $f_l'(b_l^-)$ derivatives may not be known. It is clear, however, that for any fixed k,

$$\rho_{l,k} = \rho_l(\upsilon_l) = \tfrac{1}{M_l}\sum_{r=1}^{M_l} \rho_{l,k,r}$$

$$= \tfrac{1}{2M_l}\sum_{r=1}^{M_l}\left\{ h_{l,0,k,r}\; f(a_l) \;+\; h_{l,n_l-1,k,r}\; f(b_l^-) + \sum_{i=1}^{n_l-1}(h_{l,i-1,k,r} + h_{l,i,k,r})\; f\Big[a_l + \sum_{j=0}^{i-1} h_{l,j,k,r}\Big] \right\} + \epsilon,$$

is an average over M_l random subdivisions of the I_l interval,

$$\epsilon = \overline{O}\Big[\,(h^2_{l,0,k,r}\,,\; h^2_{l,n_l-1,k,r})\,\Big] \approx -\tfrac{1}{2M_l}\sum_{r=1}^{M_l}\Big[\,h^2_{l,0,k,r} f_l'(a_l) \;-\; h^2_{l,n_l-1,k,r} f_l'(b_l^-)\,\Big] + \cdots\cdots \tag{18}$$

and $\epsilon = \overline{O}\Big[\,(h^2_{l,0,k,r}\,,\; h^2_{l,n_l-1,k,r})\,\Big] \to 0$, as n_l (or k)$\to \infty$ and $\rho_l(0) = \rho_l$.

A unique feature of the present method is that $\rho_l(0)$ shall not be determined via increasing k in a Romberg table [5] but statistically by means of a least squares regression line for a finite number N_l of $\rho_l(\upsilon_l)$ values. Obviously both $\rho_{l,k}$ of (18) and associated standard deviation

$$s_{l,k} = s_l(\upsilon_l) = \left[\sum_{r=1}^{M_l}(\rho_{l,k,r} - \rho_{l,k})^2 \,/\, (M_l - 1)\right]^{1/2} \text{ can be evaluated simultaneously.}$$

Let us assume that $\rho_l(\upsilon_l)$ follows, near $\upsilon_l = 0$, a linear regression model,

$$Y_l = \rho_l(\upsilon_l) = A_l\upsilon_l + B_l + \xi_l$$

in which ξ_l is a normally distributed random error term allowing for the variability in Y_l that cannot be explained by a linear relationship between υ_l and Y_l. Implicit hypothetical assumption that the variance σ_{ξ_l} of ξ_l is the same for all values of υ_l implies that Y_l is also a normally distributed random variable and with the same variance for all vales of υ_l. In practice however this situation is not realistic since the variance in Y_l is known from experience with Romberg tables [5] to decrease with the decrease in υ_l. This is actually our reason for designing $m = 1$ for the scatterplot $Y_{l,m} = \rho_l(\upsilon_{l,m})$ to correspond to a reasonably large enough value of k_0. Correspondingly, $Y_{l,m,r} = \rho_{l,k,r} = \rho_{l,r}(\upsilon_{l,m})$ leads to

$$s_{l,m} = s_l(\upsilon_{l,m}) = \left[\sum_{r=1}^{M_l} (Y_{l,m,r} - Y_{l,m})^2 / (M_l - 1) \right]^{1/2}, \tag{19}$$

which illustrates that σ_{ξ_l} can be estimated by $\frac{1}{N_l} \sum_{m=1}^{N_l} s_{l,m}$. Moreover only when $\mathrm{E}(\xi_l) = \overline{\xi_l} = 0$,

$$E(Y_l) = A_l\upsilon_l + B_l.$$

The estimation of A_l and B_l in the present context is a statistical process that employs an estimated least squares regression line LSRL

$$\hat{Y}_l = \varphi_l\, \upsilon_l + \phi_l \tag{20}$$

in which φ_l and ϕ_l are the sample statistics[6]:

$$\phi_l = [\sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})(Y_{l,m} - \hat{Y}_l)\,] / \sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2 \quad , \; \phi_l = \overline{Y}_l - \varphi_l\; \overline{\upsilon_l}\,, \text{ with} \tag{21}$$

$$\overline{\upsilon_l} = \frac{1}{N_l} \sum_{m=1}^{N_l} \upsilon_{l,m}, \;\; \overline{Y}_l = \frac{1}{N_l} \sum_{m=1}^{N_l} Y_{l,m}.$$

The LSRL is well known to minimize $SSE(Y_l) = \sum_{m=1}^{N_l} (Y_{l,m} - \hat{Y}_{l,m})^2 = \sum_{m=1}^{N_l} (Y_{l,m} - \varphi_l\, \upsilon_{l,m} - \phi_l)^2 \sim \xi_l$, which eventually superimposes itself on the regression error $SSR(Y_l) = \sum_{m=1}^{N_l} (\hat{Y}_{l,m} - \hat{Y}_l)^2$ to establish [6] the total error $SST(Y_l) = \sum_{m=1}^{N_l} (Y_{l,m} - \hat{Y}_l)^2$, vis $SST(Y_l) = SSE(Y_l) + SSR(Y_l)$.

If $A_l = 0$ then $\mathrm{E}(Y_l) = B_l$; in this case $\mathrm{E}(Y_l)$ does not depend on υ_l and we conclude that υ_l and Y_l are not linearly related. Alternatively, if $A_l \neq 0$, we start with a hypothesis test to determine whether $A_l = 0$. This requires an estimate of the variance of ξ_l which is $s^2_{ly} = SSE(Y_l)/(N_l - 2)$, i.e.,

$$s_{ly} = [\sum_{m=1}^{N_l} (Y_{l,m} - \hat{Y}_{l,m})^2/(N_l - 2)]^{1/2}, \tag{22}$$

would be the standard error of the estimate which has $N_l - 2$ degrees of freedom [6] because of the two parameters A_l and B_l.

3.1. **t-test**

The purpose of the t-test is to see whether the sample data set $\{\upsilon_{l,m}, Y_{l,m}\}_{m=1}^{N_l}$ is usable to conclude that $A_l \neq 0$, and the hypothesis to be tested is
$H_0 : A_l = 0 \; ; \; H_a : A_l \neq 0$.
Clearly, the sampling distribution of φ_l has a mean $\mathrm{E}(\varphi_l) = A_l$ and a standard deviation

$$\sigma_{\varphi_l} = \sigma_{\xi_l} / [\sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2]^{1/2}.$$

Because we do not know σ_{ξ_l}, we develop an estimate [6] of σ_{φ_l}, denoted s_{φ_l}, based on estimating σ_{ξ_l} with s_{ly} of (22) vis

$$s_{\varphi_l} = s_{ly} / [\sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2]^{1/2}.$$

The pertaining test statistic

$$t_l = \frac{\varphi_l - A_l}{\sigma_{\varphi_l}}$$

is assumed to follow a t-distribution with $N_l - 2$ degrees of freedom. If H_0 is true then $t_l = \frac{\varphi_l}{s_{\varphi_l}}$. Alternatively, the rejection rule at level of significance ν is : Reject H_0 if $t_l < -t_{\frac{\nu}{2}}$ or $t > t_{\frac{\nu}{2}}$. Accordingly, the confidence interval for A_l is $\varphi_l \pm t_{\frac{\nu}{2}} \, s_{\varphi_l}$.

Let us move now to evaluate the confidence interval estimate of $\mathrm{E}(Y_l)$ when extrapolated to $\upsilon_l = 0$. For any particular value of υ_l, say υ_{lo}, $\mathrm{E}(Y_{lo})$ would be the corresponding expected value of Y_l while $\hat{Y}_{lo} = \varphi_l \, \upsilon_{lo} + \phi_l$ is the corresponding estimate of $\mathrm{E}(Y_{lo})$. In general we should not expect $\hat{Y}_{lo}$ to exactly equal $\mathrm{E}(Y_{lo})$. Moreover, in order to infer about how close $\hat{Y}_{lo}$ is to the true value $\mathrm{E}(Y_{lo})$, we will have to estimate the variance of $\hat{Y}_{lo}$ or its standard deviation

$$s_{\hat{Y}_{lo}} = s_{ly} \left[\frac{1}{N_l} + (\upsilon_{lo} - \overline{\upsilon_l})^2 \Big/ \sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2 \right]^{1/2}.$$

The corresponding test statistic would be

$$t_l = \frac{\hat{Y}_{lo} - \mathrm{E}(Y_{lo})}{s_{\hat{Y}_{lo}}},$$

and the confidence interval estimate of $\mathrm{E}(Y_{lo})$ is $\hat{Y}_{lo} \pm t_{\frac{\nu}{2}} \, s_{\hat{Y}_{lo}}$. In particular application to the present integrations, $\upsilon_{lo} = 0$, $\mathrm{E}(Y_{lo}) = B_l$ and $\hat{Y}_{lo} = \phi_l$. Consequently the estimated standard deviation of ϕ_l is

$$s_{\phi_l} = \min(s_{ly}, s_{lN}) \left[\frac{1}{N_l} + \overline{\upsilon_l}^2 \Big/ \sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2 \right]^{1/2}, \qquad (23)$$

where

$$s_{lN} = \left[\sum_{r=1}^{M_l} [(\rho_{l,r}(\upsilon_{l,N}) - \rho_l(\upsilon_{l,N})]^2 / (M_l - 1) \right]^{1/2}, \quad \text{while} \qquad t_l = \frac{\phi_l - B_l}{s_{\phi_l}}, \tag{24}$$

and the confidence interval estimate of $B_l = \rho_l(0)$ is

$$\phi_l \pm t_{\frac{v}{2}} s_{\phi_l}. \tag{25}$$

This fact has been based on the assumption that s_{ϕ_l} is fixed along the υ_l line . In actual fact it is possible to assume also that $s_l\ (\upsilon_l)$ follows, near $\upsilon_l = 0$, a linear regression model,

$$Z_l = s_l(\upsilon_l) = C_l\ \upsilon_l + D_l + \zeta_l$$

in which ζ_l is a normally distributed random error of this linear model. The corresponding LSRL is $\check{Z}_l = \omega_l\ \upsilon_l + \psi_l$. Furthermore, an implicit hypothetical assumption that the variance σ_{ζ_l} of ζ_l is the same for all values of υ_l implies that Z_l of the scatterplot $Z_{l,m} = s_l(\upsilon_{l,m})$ is also a normally distributed random variable and with the same variance for all vales of $\upsilon_{l,m}$. In practice however this situation is not realistic since the variance in Z_l is known from experience with Romberg tables [5] to decrease with the decrease in υ_l. Here also, only when $\mathrm{E}(\zeta_l) = \overline{\zeta_l} = 0$,

$$E(Z_l) = C_l\upsilon_l + D_l\ ,$$

and the corresponding statistic is

$$\mathfrak{t}_l = \frac{\psi_l - D_l}{s_{\psi_l}}.$$

Theorem 3.1. *Starting from a sufficiently large value of k_0 for $m = k - k_0 + 1$, if ϕ_l is the Richardson extrapolated value for $\rho_l(0)$, then the v confidence interval for ϕ_l has a lower bound of*

$$\phi_l \pm t_{\frac{v}{2}} \min(s_{\varphi_l}, \psi_l \pm \mathfrak{t}_{\frac{v}{2}} s_{\psi_l}) \tag{26}$$

and an upper bound of

$$\phi_l \pm t_{\frac{v}{2}} \max(s_{\varphi_l}, \psi_l \pm \mathfrak{t}_{\frac{v}{2}} s_{\psi_l}), \tag{27}$$

where

$$s_{\psi_l} = [\sum_{m=1}^{N_l} (Z_{l,m} - \check{Z}_{l,m})^2/(N_l - 2)]^{1/2} \left[\frac{1}{N_l} + \overline{\upsilon_l}^2 / \sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2 \right]^{1/2}. \tag{28}$$

Proof. The LSRL for $Z_{l,m} = s_l\ (\upsilon_{l,m}), \forall m$ has

$$\omega_l = [\sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})(Z_{l,m} - \overline{Z_l})\ ] / \sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2 \quad , \ \psi_l = \overline{Z_l} - \omega_l\ \overline{\upsilon_l},$$

with

$$\overline{\upsilon_l} = \frac{1}{N_l} \sum_{m=1}^{N_l} \upsilon_{l,m}\ , \quad \overline{Z_l} = \frac{1}{N_l} \sum_{m=1}^{N_l} Z_{l,m}\ .$$

Here $SSE(Z_l) = \sum_{m=1}^{N_l} (Z_{l,m} - \check{Z}_m)^2 = \sum_{m=1}^{N_l} (Z_{l,m} - \omega_l \upsilon_{l,m} - \psi_l)^2 \sim \zeta_l$ and

$$s_{lz}^2 = SSE(Z_l)/(N_l - 2)\ . \tag{29}$$

3.2. **Second t-test**

Before k_0, it is assumed that the pertaining points in the scatterplot are outliers to the regression process. Now if $C_l = 0$ then $\mathrm{E}(Z_l) = D_l$; then υ_l and Z_l are not linearly related. Alternatively, if $C_l \neq 0$, we start with a hypothesis test to determine whether $C_l = 0$. This requires an estimate of the variance σ_{ζ_l} and a hypothesis t-test,
$H_0 : C_l = 0\ ;\ H_a : C_l \neq 0$,
is needed again here to see whether $\{\upsilon_{l,m}, Z_{l,m}\}_{m=1}^{N_l}$ is usable.
The sampling distribution of ω_l has $\mathrm{E}(\omega_l) = C_l$ and the standard deviation

$$\sigma_{\omega_l} = \sigma_{\xi_l}/[\sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2]^{1/2}.$$

Since σ_{ξ_l} is unknown, our estimate of σ_{ω_l}, denoted s_{ω_l} ,based on estimating σ_{ξ_l} with s_{lz} of (29) will be

$$s_{\omega_l} = s_{lz}/[\sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2]^{1/2}.$$

The pertaining test statistic

$$ţ_l = \frac{\omega_l - C_l}{s_{\omega_l}}$$

is assumed, like t_l, to follow a t-distribution with $N_l - 2$ degrees of freedom. If H_0 is true then $ţ_l = \frac{\omega_l}{s_{\omega_l}}$. Alternatively, the rejection rule at level of significance ν is : Reject H_0 if $ţ < -ţ_{\frac{\nu}{2}}$ or $ţ > ţ_{\frac{\nu}{2}}$. Accordingly, the confidence interval for C_l is $\omega_l \pm ţ_{\frac{\nu}{2}}\ s_{\omega_l}$.
For any particular value of υ_l, say υ_{lo}, $\mathrm{E}(Z_{lo})$ would be the corresponding expected value of Z_l while $\check{Z}_{lo} = \omega_l \upsilon_{lo} + \psi_l$ is the corresponding estimate of $\mathrm{E}(Z_{lo})$. For inference about how close $\check{Z}_{lo}$ is to the true value $\mathrm{E}(Z_{lo})$, we will have to estimate the variance of $\check{Z}_{lo}$ and of its standard deviation

$$s_{\check{Z}_{lo}} = s_{lz}\left[\frac{1}{N_l} + (\upsilon_{lo} - \overline{\upsilon})^2 \Big/ \sum_{m=1}^{N_l} (\upsilon_{l,m} - \overline{\upsilon_l})^2 \right]^{1/2}.$$

The corresponding test statistic is

$$ţ_1 = \frac{\check{Z}_{lo} - \mathrm{E}(Z_{lo})}{s_{\check{Z}_{lo}}},$$

and the confidence interval estimate of $\mathrm{E}(Z_{lo})$ is $\check{Z}_{lo} \pm ţ_{\frac{\nu}{2}}\ s_{\check{Z}_{lo}}$. In particular application to the present integrations $\upsilon_{lo} = 0$, $E(Z_{lo}) = D_l$ and $\check{Z}_{lo} = \psi_l$. Consequently, the confidence interval estimate of $D_l = s_l(0)$ is $\psi_l \pm t_{\frac{\nu}{2}}\, s_{\psi_l}$. Consideration finally of this result in (25)completes the proof. ■

4. Enhanced Randomized Romberg Integration over I_l

Enhancement of the accuracy of evaluating the ρ_l-integral, for a monotonically increasing or decreasing function over $I_l = [a_l, b_l^-]$, is conceived to take place as a result of application of (13) of Proposition 1, which calls for a Romberg integration only of the $\frac{\gamma_l}{\rho_l}(\upsilon_l)$ ratio. When evaluating the pertaining γ_l –integral, we first consider random subdivisions of the $J_l = [\alpha_l, \beta_l]$ interval according to

$$\tau_{i-1,k,r} + \delta_{l,i-1,k,r} = \tau_{i,k,r},$$

and follow the same steps taken earlier with $f_l(x)$ over$[a_l, b_l^-]$ to arrive at

$$\gamma_{l,k,r} = \tfrac{1}{2}\left\{\delta_{l,0,k,r}\eta(\alpha_l) + \delta_{l,n_l-1,k,r}\,\eta(\beta_l) + \sum_{i=1}^{n_l-1}(\delta_{l,i-1,k,r} + \delta_{l,i,k,r})\,\eta\,[\alpha_l + \sum_{j=0}^{i-1}\delta_{l,j,k,r}]\right\} + O\left[(\delta^2_{l,0,k,r},\ \delta^2_{l,n_l-1,k,r})\right],$$

$$\gamma_{l,k} = \tfrac{1}{M_l}\sum_{r=1}^{M_l}\gamma_{l,k,r}$$

$$= \tfrac{1}{2M_l}\sum_{r=1}^{M_l}\left\{\delta_{l,0,k,r}\,\eta(\alpha_l) + \delta_{l,n_l-1,k,r}\,\eta(\beta_l) + \sum_{i=1}^{n_l-1}(\delta_{l,i-1,k,r} + \delta_{l,i,k,r})\,\eta\,[\alpha_l + \sum_{j=0}^{i-1}\delta_{l,j,k,r}]\right\} + \varepsilon, \qquad (30)$$

$$\varepsilon = \overline{O}\left[(\delta^2_{l,0,k,r},\ \delta^2_{l,n_l-1,k,r})\right] = -\tfrac{1}{12M_l}\sum_{r=1}^{M_l}\left[\delta^2_{l,0,k,r}\,\eta'(\alpha_l) - \delta^2_{l,n_l-1,k,r}\,\eta'(\beta_l)\right] + \cdots\cdots$$

Clearly $\gamma_{l,k}$ is an average of M_l values of $\gamma_{l,k,r}$ and the associated standard deviation is

$$s_{lk} = \left[\sum_{r=1}^{M_l}(\gamma_{l,k,r} - \gamma_{l,k})^2 / (M_l - 1)\right]^{1/2}.$$

For reasons of computational economy we shall mean to use the same discretization data of $f_l(x)$ over $[a_l, b_l^-]$ used in the evaluation of the ρ_l –integral.

Proposition 4.1. *Let*

$$g_{l,i,k,r} = f_l\,(x_{i+1,k,r}) - f_l\,(x_{i,k,r}), \qquad (31)$$

then for $i = 1,2,3,\ldots\ldots, n_l = 2^{k-1}$ *and* $\forall$ *k and r the following holds*

$$\eta_l(\tau_{i,k,r}) = \tfrac{1}{\sqrt{2}}\left\{f\,[a_l + \sum_{j=0}^{i-1}h_{l,j,k,r}] - \sum_{j=0}^{i-1}h_{l,j,k,r} - a_l\right\}. \qquad (32)$$

Proof. For any point $(x_{i,k,r}\ ,\ f_l(x_{i,k,r}))$on the $x - y$ plane, the corresponding point $(\tau_{i,k,r}\ ,\eta_l(\tau_{i,k,r}))$ on the $\tau - z$ plane can be shown graphically to satisfy

$$\eta_l(\tau_{i,k,r}) = \frac{1}{\sqrt{2}}[f_l(x_{i,k,r}) - x_{i,k,r}].$$

As
$$\delta_{i,k,r} = \tfrac{1}{\sqrt{2}}\left(h_{i,k,r} + g_{i,k,r}\right), \tag{33}$$
the affine map (4) leads directly to
$$\tau_{i,k,r} - \tau_{i-1,k,r} = \tfrac{1}{\sqrt{2}}\left\{[x_{i,k,r} - x_{i-1,k,r}] + [f_l(x_{i,k,r}) - f_l(x_{i-1,k,r})]\right\},$$
from which relations(31)-(32) directly follow. ■

Making use now of this proposition in (30) yields

$$\begin{aligned}\gamma_{l,k} &= \tfrac{1}{M_l}\sum_{r=1}^{M_l}\gamma_{l,k,r} = \tfrac{1}{4M_l}\sum_{r=1}^{M_l}\Big\{[h_{l,0,k,r} + f_l(a_l + h_{l,0,k,r}) - f_l(a_l)][f_l(a_l) - a_l] \\ &+[f_l(b_l^-) - b_l^-]\left[h_{l,n_l-1,k,r} - f_l[a_l + \sum_{j=0}^{n_l-2} h_{l,j,k,r}] + f_l(b_l^-)\right] \\ &+\sum_{i=1}^{n_l-1}\{h_{l,i-1,k,r} + h_{l,i,k,r} + f_l\,[a_l + \sum_{j=0}^{i} h_{l,j,k,r}] \\ &- f_l[a_l + \sum_{j=0}^{i-2} h_{l,j,k,r}]\}\{f_l\,[a_l + \sum_{j=0}^{i-1} h_{l,j,k,r}] - \sum_{j=0}^{i-1} h_{l,j,k,r} - a_l\}\Big\} + \varepsilon\end{aligned} \tag{34}$$

in terms of the same discretized data for $\rho(\upsilon_l)$. The symmetrized-to-conventional integral ratio is $\frac{\gamma_{l,k}}{\rho_{l,k}} = \frac{\gamma_l}{\rho_l}(\upsilon_l)$ with $\gamma_{l,k}$ of (34) , $\rho_{l,k}$ of (18) and associated standard deviation

$$s_{lk} = s_l(\upsilon_l) = \left[\sum_{r=1}^{M_l}\left(\frac{\gamma_{l,k,r}}{\rho_{l,k,r}} - \frac{\gamma_{l,k}}{\rho_{l,k}}\right)^2 / (M_l - 1)\right]^{1/2}$$

can all simultaneously be evaluated. Richardson extrapolation can be applied as before to predict ϕ_l of $\frac{\gamma_l}{\rho_l}(0)$ with the scatterplot $Y_{l,m} = \frac{\gamma_l}{\rho_l}(\upsilon_{l,m})$, for which Theorem 3.1 holds at it stands. This is eventually used in the enhanced estimation of $\rho_l(0)$ by means of

$$\rho_l(0) = (\text{Ç}_{b_l} - \text{Ç}_{a_l})/[\frac{\gamma_l}{\rho_l}(0) - 1].$$

Corollary 4.1. *The Riemann integral over I of piecewise continous $f(x)$ is the following indirect sum*

$$\rho = \sum_{l=1}^{L}\rho_l(0) = \sum_{l=1}^{L}(\text{Ç}_{b_l} - \text{Ç}_{a_l})/[\frac{\gamma_l}{\rho_l}(0) - 1],$$

over γ_l symetrizable integrals, and it is not symmetrizable itself, i.e.,

$$\gamma \neq \sum_{l=1}^{L} \gamma_l(0).$$

Proof. Can be made graphically for $f(x)$ of (1) over $I = [a, b]$. Indeed each γ_l integral is defined over a certain $J_l = [\alpha_l, \beta_l]$ but the set $\{J_l\}_{l=1}^{L}$ is not a partition of a single larger set ; which makes the concept of a γ is meaningless. ■

5. Conclusions

The Romberg method, based on the trapezoid rule, is demonstrated in this work to be ideal for the computation of the symmetrized integral or the symmetrized-to-conventional integral ratios. Randomization in Romberg integration and its enhancement, by means of the principle of Corollary 4.2, indicates a tendency to produce Richardson extrapolation regression lines with relatively elevated correlation coefficients. The comparison is with the fixed-step conventional Romberg method having the same number of subdivisions. Statistical error bounds have been established, for any desired level of significance, for this new computationally economic method of integration.

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