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General Class of Nonlinear Implicit Inclusion Problems and A – Maximal Relaxed Monotone Models

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Abstract. *Some existence theorems in the context of solving a general class of nonlinear implicit inclusion problems involving A –maximal relaxed monotone mappings are established. The solvability of the problems of this form is much dependent on the generalized resolvent operator technique under the framework of A –maximal relaxed monotonicity (and H –maximal monotonicity).*

Key words: Implicit variational inclusions, Maximal monotone mapping, A – maximal relaxed monotone mapping, Generalized resolvent operator.

AMS Subject Classifications : 49J40, 65B05

1. Introduction

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle .,. \rangle$. We consider the implicit inclusion problem: determine a solution $u \in X$ such that

$$0 \in A(u) + M(g(u)), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on $X,$ and $A, g : X \to X$ are single-valued mappings on X with $range(g) \cap dom(M) \neq \emptyset.$
For $g = I,$ this inclusion problem reduces to: find a solution to

$$0 \in A(u) + M(u). \tag{2}$$

When $A = I,$ (1) reduces to: determine a solution $u \in X$ such that

$$0 \in I(u) + M(g(u)), \tag{3}$$

where $g : X \to X$ is a single-valued mapping, and $M : X \to 2^X$ is a set-valued mapping on X with $range(g) \cap dom(M) \neq \emptyset.$

The notion of A –maximal monotonicity was introduced by the author [7], while investigating the solutions of variational inclusion problems using the resolvent operator technique, that generalizes the existing general theory of maximal monotone operators, including the H–maximal monotonicity by Fang and Huang [2]. We intend in this paper to generalize the existence theorems based on the notion of A–maximal relaxed monotonicity, and as a result we apply a generalized resolvent operator technique. The obtained results are general in nature. This model has also applications to nonlinear evolution equations/inclusions using the Yosida approximations. The generalized resolvent operator techniques can also be applied to several other fields, for stance, global optimization and control theory, operations research, mathematical finance, management and decision sciences, and mathematical programming. For more literature, we recommend the reader [1- 19].

2. A-Preliminaries

In this section we discuss some results based on the basic properties and auxiliary results on A– maximal relaxed monotonicity (also referred to as A– monotonicity in literature) and its variant forms. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, that is, the set $\{(x,y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x,y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x,y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$dom(M) = \{x \in X : \exists y \in X : (x,y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

dom(M)=X, shall denote the full domain of M, and the range of M is defined by

$$R(M) = \{y \in X : \exists x \in X : (x,y) \in M\}.$$

The inverse M^{-1} of M is $\{(y,x) : (x,y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{x, \rho y) : (x,y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y+z) : (x,y) \in L, (x,z) \in M\}.$$

Definition 2.1. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:
(i) (r) – strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2 \; \forall \, (u,u^*), (v,v^*) \in graph(M).$$

(ii) (m) –relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2 \; \forall \, (u,u^*), (v,v^*) \in graph(M).$$

(iii) (s) –Lipschitz continuous if there exists a positive constant s such that
$\|u^* - v^*\| \leq s\|u - v\| \; \forall \, (u,u^*), (v,v^*) \in graph(M)$.

Definition 2.2. [7] Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be A– maximal (m) –relaxed monotone if
(i) M is (m) –relaxed monotone for $m > 0$.
(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.3. [7] Let $A : X \to X$ be an (r) –strongly monotone mapping and let $M : X \to 2^X$

be an A –maximal (m) –relaxed monotone mapping. Then the generalized resolvent operator $J^M_{\rho,A} : X \to X$ is defined by

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Definition 2.4. [2] Let $H : X \to X$ be (r) – strongly monotone. The map $M : X \to 2^X$ is said to be to H – maximal monotone if

(i) M is monotone,

(ii) $R(H + \rho M) = X$ for $\rho > 0$.

Definition 2.5. [2] Let $H : X \to X$ be an (r) –strongly monotone mapping and let $M : X \to 2^X$ be an H – monotone mapping. Then the generalized resolvent operator $J^M_{\rho,H} : X \to X$ is defined by

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u).$$

Proposition 2.1. [7] *Let $A : X \to X$ be a (r) –strongly monotone single-valued mapping and let $M : X \to 2^X$ be an A – maximal (m) –relaxed monotone mapping. Then $(A + \rho M)$ is maximal monotone for $\rho > 0$.*

Proposition 2.2. [7] *Let $A : X \to X$ be an (r) –strongly monotone mapping and let $M : X \to 2^X$ be an A – maximal (m) –relaxed monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.*

Proposition 2.3. [1] *Let $H : X \to X$ be a (r) –strongly monotone single-valued mapping and let $M : X \to 2^X$ be an H – maximal monotone mapping. Then $(H + \rho M)$ is maximal monotone for $\rho > 0$.*

Proposition 2.4. *Let $H : X \to X$ be an (r) –strongly monotone mapping and let $M : X \to 2^X$ be an H –- maximal monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.*

3. A-Maximal Relaxed Monotonicity and Existence Theorems

This section deals with the existence theorems on the solvability of the inclusion problem (1) based on the A-maximal relaxed monotonicity.

Lemma 3.1. [7] *Let X be a real Hilbert space, let $A : X \to X$ be (r) –strongly monotone, and let $M : X \to 2^X$ be A – maximal (m) –relaxed monotone. Then the generalized resolvent operator associated with M and defined by*

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u)\, \forall\, u \in X,$$

is $(\frac{1}{r-\rho m})$ – Lipschitz continuous for $r - \rho m > 0$,

that is,

$$\|J^M_{\rho,A}(u) - J^M_{\rho,A}(v)\| \leq \frac{1}{r - \rho m}\|u - v\|, \tag{4}$$

where $r - \rho m > 0$.

Lemma 3.2. [2] *Let X be a real Hilbert space, let $H : X \to X$ be (r) –strongly monotone, and* *let* $M : X \to 2^X$ *be* $H-$ *maximal monotone. Then the generalized resolvent operator associated with* M *and defined by*

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u)\, \forall\, u \in X,$$

is $(\frac{1}{r})-$ *Lipschitz continuous,*
that is,

$$\|J^M_{\rho,H}(u) - J^M_{\rho,H}(v)\| \leq \frac{1}{r}\|u - v\|. \tag{5}$$

Theorem 3.1. *Let* X *be a real Hilbert space, let* $A : X \to X$ *be* $(r)-$*strongly monotone, and let* $M : X \to 2^X$ *be* $A-$ *maximal* $(m)-$*relaxed monotone. Let* $g : X \to X$ *be a map on* X*. Then the following statements are equivalent:*

(i) *An element* $u \in X$ *is a solution to* (1).

(ii) *For an* $u \in X$*, we have*

$$g(u) = J^M_{\rho,A}(A(g(u)) - \rho A(u)),$$

where

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Theorem 3.2. *Let* X *be a real Hilbert space, let* $H : X \to X$ *be* $(r)-$*strongly monotone, and let* $M : X \to 2^X$ *be* $H-$ *maximal monotone. Let* $g : X \to X$ *be a map on* X*. Then the following statements are equivalent:*

(i) *An element* $u \in X$ *is a solution to* (1).

(ii) *For an* $u \in X$*, we have*

$$g(u) = J^M_{\rho,H}(H(g(u)) - \rho H(u)),$$

where

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u).$$

Theorem 3.3. *Let* X *be a real Hilbert space, and let* $M : X \to 2^X$ *be maximal monotone. Let* $g : X \to X$ *be a map on* X*. Then the following statements are equivalent:*

(i) *An element* $u \in X$ *is a solution to* (3).

(ii) *For an* $u \in X$*, we have*

$$g(u) = J^M_{\rho}((g(u) - \rho u),$$

where

$$J^M_{\rho}(u) = (I + \rho M)^{-1}(u).$$

Lemma 3.3. *Let* X *be a real Hilbert space, let* $A : X \to X$ *be* $(r)-$*strongly monotone and* $(s)-$ *Lipschitz continuous, and let* $M : X \to 2^X$ *be* A*- maximal* $(m)-$*relaxed monotone. Then*

$$\langle (J^M_{\rho,A} o A)(u) - (J^M_{\rho,A} o A)(v), u - v\rangle \leq \frac{s}{r - \rho m}\|u - v\|^2\, \forall u, v \in X.$$

Lemma 3.4. *Let X be a real Hilbert space, let $H : X \to X$ be (r) –strongly monotone and (s) –Lipschitz continuous, and let $M : X \to 2^X$ be H –maximal monotone. Then*

$$\langle (J^M_{\rho,H} oH)(u) - (J^M_{\rho,H} oH)(v), u - v\rangle \leq \frac{s}{r}\|u-v\|^2\ \forall u,v \in X.$$

Theorem 3.4. *Let X be a real Hilbert space, let $A : X \to X$ be (r) –strongly monotone and (s) –Lipschitz continuous, and let $M : X \to 2^X$ be A –maximal (m) –relaxed monotone. Let $g : X \to X$ be (t) –strongly monotone and (β) –Lipschitz continuous. Then (1) has a unique solution $x^* \in X$ for*

$$|\rho - \frac{r}{s^2}| < \frac{\sqrt{r^2 - s^2(a+b)(2-(a+b))}}{s^2},$$

$$r > s\sqrt{(a+b)(2-(a+b))},\ a = 2\sqrt{1-2t+\beta^2} < 1,$$

and $b = \sqrt{\beta^2 - 2rt^2 + s^2\beta^2} < 1.$

Proof. First we define a function $G : X \to X$ by

$$G(u) = u - g(u) + J^M_{\rho,A}(A(g(u)) - \rho A(u)),$$

and then to show that G is contractive.
Applying Lemma 3.1, we have

$$\begin{aligned}
&\|G(u) - G(v)\| \\
&= \|u - v - (g(u) - g(v)) + J^M_{\rho,A}(A(g(u)) - \rho A(u)) - J^M_{\rho,A}(A(g(v)) - \rho A(v))\| \\
&\leq \|u - v - (g(u) - g(v))\| + \tfrac{1}{r-\rho m}\|A(g(u)) - A(g(v)) - \rho(A(u) - A(v))\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|A(g(u)) - A(g(v)) - \rho(A(u) - A(v))\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|A(g(u)) - A(g(v)) - (u-v)\| \\
&+ \|u - v - \rho(A(u) - A(v))\| \\
&\leq 2\|u - v - (g(u) - g(v))\| + \|A(g(u)) - A(g(v)) - (g(u) - g(v))\| \\
&+ |u - v - \rho(A(u) - A(v))\|, \qquad (6)
\end{aligned}$$

where $r - \rho m > 1$.
Since g is (t) –strongly monotone and β –Lipschitz continuous, we have

$$\|u - v - (g(u) - g(v))\| \leq \sqrt{1 - 2t + \beta^2}\,\|u - v\|. \qquad (7)$$

Similarly, using the strong monotonicity and Lipschitz continuity assumptions on A and $g,$ we find

$$\|A(g(u)) - A(g(v)) - (g(u) - g(v))\| \leq \sqrt{\beta^2 - 2rt^2 + s^2\beta^2}\,\|u - v\|, \qquad (8)$$

$$\|u - v - \rho(A(u) - A(v))\| \leq \sqrt{1 - 2\rho r + \rho^2 s^2}\,\|u - v\|. \qquad (9)$$

In light of (6) – (9) we have

$$\|G(u) - G(v)\| \leq \theta\|u - v\|, \qquad (10)$$

where

$$\theta = 2\sqrt{1-2t+\beta^2} + \sqrt{\beta^2-2rt^2+s^2\beta^2} + \sqrt{1-2\rho r+\rho^2 s^2} < 1$$

for

$$|\rho - \frac{r}{s^2}| < \frac{\sqrt{r^2-s^2(a+b)(2-(a+b))}}{s^2},$$

$$r > s\sqrt{(a+b)(2-(a+b))}, a = \sqrt{1-2t+\beta^2} < 1,$$

$b = \sqrt{\beta^2-2rt^2+s^2\beta^2} < 1,$ and $r-\rho m > 1.$

Hence, G is a contractive mapping. ■

Theorem 3.5. *Let X be a real Hilbert space, let $H : X \to X$ be (r) –strongly monotone and (s) –Lipschitz continuous, and let $M : X \to 2^X$ be $H-$ maximal monotone. Let $g : X \to X$ be (t) –strongly monotone and (β) –Lipschitz continuous. Then (1) has a unique solution $x^* \in X$ for*

$$|\rho - \frac{r}{s^2}| < \frac{\sqrt{r^2-s^2[1-(1-(a^*+b^*))^2 r^2]}}{s^2},$$

$$r > s\sqrt{[1-(1-(a^*+b^*))^2 r^2]}, a^* = 2\sqrt{1-2t+\beta^2} < 1,$$

$b^* = \frac{1}{r}\sqrt{\beta^2-2rt^2+s^2\beta^2} < 1,$ *and* $r > 1.$

Proof. Although the proof is similar to that of Theorem 3.4, we include a brief sketch because it is less restrictive in the sense of boundary conditions imposed in Theorem 3.4. First we define a function $G : X \to X$ by

$$G(u) = u - g(u) + J^M_{\rho_k,H}(H(g(u)) - \rho H(u)),$$

and then to show that G is contractive.
Applying Lemma 3.2, we have

$$\begin{aligned}
&\|G(u)-G(v)\| \\
&= \|u-v-(g(u)-g(v)) + J^M_{\rho,H}(H(g(u)) - \rho H(u)) - J^M_{\rho,H}(H(g(v)) - \rho H(v))\| \\
&\leq \|u-v-(g(u)-g(v))\| + \tfrac{1}{r}\|H(g(u)) - H(g(v)) - \rho(H(u)-H(v))\| \\
&\leq \|u-v-(g(u)-g(v))\| + \tfrac{1}{r}\|H(g(u)) - H(g(v)) - \rho(H(u)-H(v))\| \\
&\leq \|u-v-(g(u)-g(v))\| + \tfrac{1}{r}\|H(g(u)) - H(g(v)) - (u-v)\| \\
&+\tfrac{1}{r}\|u-v-\rho(H(u)-H(v))\| \\
&\leq 2\|u-v-(g(u)-g(v))\| + \|H(g(u)) - H(g(v)) - (g(u)-g(v))\| \\
&\quad + \|u-v-\rho(H(u)-H(v))\|,
\end{aligned} \tag{11}$$

where $r > 1.$ Since g is (t) –strongly monotone and β –Lipschitz continuous, we have

$$\|u-v-(g(u)-g(v))\| \leq \sqrt{1-2t+\beta^2}\,\|u-v\|. \tag{12}$$

Similarly, using the strong monotonicity and Lipschitz continuity assumptions on H and $g,$ we find

$$\|H(g(u)) - H(g(v)) - (g(u)-g(v))\| \leq \sqrt{\beta^2-2rt^2+s^2\beta^2}\,\|u-v\|, \tag{13}$$

$$\|u - v - \rho(H(u) - H(v))\| \leq \sqrt{1 - 2\rho r + \rho^2 s^2}\, \|u - v\|. \tag{14}$$

In light of $(12) - (14)$, we have

$$\|G(u) - G(v)\| \leq \theta^* \|u - v\|, \tag{15}$$

where

$$\theta^* = 2\sqrt{1 - 2t + \beta^2} + \frac{1}{r}\sqrt{\beta^2 - 2rt^2 + s^2\beta^2} + \sqrt{1 - 2\rho r + \rho^2 s^2} < 1$$

for

$$|\rho - \frac{r}{s^2}| < \frac{\sqrt{r^2 - s^2[1 - (1 - (a^* + b^*))^2 r^2]}}{s^2},$$

$$r > s\sqrt{[1 - (1 - (a^* + b^*))^2 r^2]},\ a^* = 2\sqrt{1 - 2t + \beta^2} < 1,$$

and $b^* = \sqrt{\beta^2 - 2rt^2 + s^2\beta^2} < 1$.

Hence, G is a contractive mapping. ■

We note that when $A = I$, the identity mapping in Theorem 3.4, we have:

Theorem 3.6. *Let X be a real Hilbert space, and let* $M : X \to 2^X$ *be maximal monotone. Let* $g : X \to X$ *be* (t) *–strongly monotone and* (β) *–Lipschitz continuous. Then (3), that is,*

$$0 \in I(u) + M(g(u)), \tag{16}$$

where $M : X \to 2^X$ *is a set-valued mapping on* X, *and* $g : X \to X$ *is a single-valued mapping on* X *with* $range(g) \cap dom(M) \neq \emptyset$, *has a unique solution* $x^* \in X$ *for* $|\rho - 1| < (1 - a)$, *where* $a = 2\sqrt{1 - 2t + \beta^2} < 1$.

Proof. First we define a function $G : X \to X$ by

$$G(u) = u - g(u) + J_\rho^M(g(u) - \rho A(u)),$$

where $J_\rho^M(u) = (I + \rho M)^{-1}(u)$, and then to show that G is contractive.

Since J_ρ^M is nonexpansive, we have

$$\begin{aligned}
&\|G(u) - G(v)\| \\
&= \|u - v - (g(u) - g(v)) + J_\rho^M(g(u) - \rho u) - J_\rho^M(g(v) - \rho v)\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|g(u) - g(v) - \rho(u - v)\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|g(u) - g(v) - \rho(u - v))\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|g(u) - g(v) - (u - v)\| \\
&+ \|u - v - \rho(u - v)\| \\
&\leq 2\|u - v - (g(u) - g(v))\| + |u - v - \rho(A(u) - A(v))\|. \qquad (17)
\end{aligned}$$

As given that g is (t) –strongly monotone and β –Lipschitz continuous, we have

$$\|u - v - (g(u) - g(v))\| \leq \sqrt{1 - 2t + \beta^2}\, \|u - v\|. \tag{18}$$

Next, we have

$$\|u - v - \rho(u - v)\| = (1 - \rho)\|u - v\|. \tag{19}$$

Thus, we have

$$\|G(u) - G(v)\| \leq \theta\|u - v\|, \tag{20}$$

where

$\theta = 2\sqrt{1 - 2t + \beta^2} + \rho - 1 < 1.$ ■

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