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Anticipated Backward Doubly Stochastic Differential Equations With Stochastic Lipschitzian Coefficients*

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Abstract. *In this work, we study the anticipated backward doubly stochastic differential equations (ABDSDEs) whose coefficients satisfy a stochastic Lipschitz condition. An existence and uniqueness theorem is formulated and proved for this equation. A comparison theorem for this type of ABDSDEs is also reported.*

Key words : Anticipated Backward Doubly Stochastic Differential Equations, Stochastic Lipschitz Conditions, Itô's Representation Formula, Gronwall Lemma.

AMS Subject Classifications : 60H05, 60H10, 60H20

1. Introduction

Nonlinear Backward stochastic differential equations (BSDEs in short) have first been introduced by Pardoux and Peng [3] in order to give a probabilistic interpretation for the solutions to semi-linear partial differential equations (PDEs). Since then, and in recent years, BSDEs have received considerable attention due to their wide applicability in a number of different areas such as financial mathematics and partial differential equations. After their formulation of the theory of BSDEs, Pardoux and Peng considered in [4] a new kind of BSDEs, which is a class of backward doubly stochastic differential equations (BDSDEs) with two different directions for the stochastic integrals. On one hand, they proved existence and uniqueness of solutions to BDSDEs under uniform Lipschitz conditions on the coefficients. Then Owo [2] discussed BDSDEs with a stochastic Lipschitz condition and proved the existence and uniqueness of their solution.

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On the other hand, recently, Xu [6] studied the anticipated backward doubly stochastic differential equations (ABDSDEs) of the following form:

$$\begin{cases} -dY_t = f(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)})dt + g(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)})dB_t - Z_t dW_t, \\ 0 \leq t \leq T, \\ Y_t = \xi_t, \quad Z_t = \eta_t \qquad T \leq t \leq T+K, \end{cases}$$

where $\xi_.$, $\eta_.$ are given stochastic processes, and $\delta(.)$, $\zeta(.)$ are given nonnegative deterministic functions. In [6], the author formulated an existence and uniqueness result under Lipschitz conditions, a comparison theorem for one-dimensional ABDSDEs and solved a stochastic control problem using the duality between linear stochastic differential delay equations and ABDSDEs.

In the present paper, we deal with a class of ABDSDEs under stochastic Lipschitz conditions. Inspired by the works of Owo [2] and Xu [6], we prove that under stochastic Lipschitz conditions, the solution of these ABDSDEs exists uniquely. The key point is an iterated scheme on a suitable sequence. Based on [7], we establish a comparison theorem for one dimensional ABDSDEs.

The paper is organized as follows. In section 2, we introduce some preliminaries. In section 3, we establish the existence and uniqueness of a solution for the ABDSDEs in the case of a stochastic Lipschitz condition. A comparison theorem for the solutions of ABDSDEs is reported in section 4.

2. Preliminaries

Let Ω be a non-empty set, $\mathcal{F}$ a σ –algebra of sets of Ω and $\mathbf{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ defines a probability space, which is assumed to be complete. For a fixed real $0 < T \leq \infty$, we assume that we are given two mutually independent processes:

- a ℓ –dimensional Brownian motion $(B_t)_{0 \leq t \leq T}$,
- a d –dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$.

Then we consider the family $(\mathcal{F}_t)_{0 \leq t \leq T}$ given by

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \qquad 0 \leq t \leq T, \qquad G_s = \mathcal{F}_{0,s}^W \vee \mathcal{F}_{s,T+K}^B, \quad 0 \leq s \leq T+K,$$

where for any process $\{\phi_t\}_{t \geq 0}$, $\mathcal{F}_{s,t}^\phi = \sigma\{\phi_r - \phi_s, \ s \leq r \leq t\} \vee N$, $\mathcal{F}_t^\phi = \mathcal{F}_{0,t}^\phi$. N denotes the class of $\mathbf{P}$ –null sets of $\mathcal{F}$. Note that $(\mathcal{F}_t)_{0 \leq t \leq T}$ does not constitute a classical filtration.

For every random process $(a(t))_{t \geq 0}$ with positive values, such that $a(t)$ is G_t –measurable for any $t \geq 0$, we define an increasing process $(A(t))_{t \geq 0}$ by setting $A(t) = \int_0^t a(s)ds$.

For $k \in \mathbf{N}^*$ and $\beta > 0$, we consider the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$):

- $L^2(\beta, G_T, \mathbf{R}^k)$ the space of G_T-measurable random variables such that
$$\mathbf{E}\left[e^{\beta A(T)} |\xi_T|^2 \right] < +\infty;$$

- $S^2_{[0,T]}(\beta, G, \mathbf{R}^k)$ the space of G_t –adapted càdlàg processes:

$$\Psi : [0,T] \times \Omega \to \mathbf{R}^k, \quad \|\Psi\|^2_{S^2(\beta,\mathbf{R}^k)} = \mathbf{E}\left(\sup_{0 \le t \le T} e^{\beta A(t)} |\Psi_t|^2 \right) < +\infty;$$

- $M^2_{[0,T]}(\beta, a, G, \mathbf{R}^k)$ the space of G_t –progressively measurable processes:

$$\Psi : [0,T] \times \Omega \to \mathbf{R}^k, \quad \|\Psi\|^2_{M^2(\beta,a,\mathbf{R}^k)} = \mathbf{E} \int_0^T e^{\beta A(t)} a(t) |\Psi_t|^2 \, dt < +\infty;$$

- $M^2_{[0,T]}(\beta, G, \mathbf{R}^{k\times d})$ the space of G_t –progressively measurable processes

$$\Psi : [0,T] \times \Omega \to \mathbf{R}^{k\times d}, \quad \|\Psi\|^2_{M^2(\beta,\mathbf{R}^{k\times d})} = \mathbf{E} \int_0^T e^{\beta A(t)} |\Psi_t|^2 \, dt < +\infty;$$

- $C^2_G(\beta, a, [0,T], \mathbf{R}^k) = M^2_{[0,T]}(\beta, a, G, \mathbf{R}^k) \times {}^2_{[0,T]}(\beta, G, \mathbf{R}^{k\times d})$ endowed with the norm

$$\|(Y,Z)\|^2_{C^2_G(\beta,a,T,\mathbf{R}^k)} = \|Y\|^2_{M^2(\beta,a,\mathbf{R}^k)} + \|Z\|^2_{M^2(\beta,\mathbf{R}^{k\times d})};$$

- $B^2_G(\beta, a, [0,T], \mathbf{R}^k) = \left(M^2_{[0,T]}(\beta, a, G, \mathbf{R}^k) \cap S^2_{[0,T]}(\beta, G, \mathbf{R}^k) \right) \times M^2_{[0,T]}(\beta, G, \mathbf{R}^{k\times d})$

endowed with the norm

$$\|(Y,Z)\|^2_{B^2_G(\beta,a,T,\mathbf{R}^k)} = \|Y\|^2_{S^2(\beta,\mathbf{R}^k)} + \|Y\|^2_{M^2(\beta,a,\mathbf{R}^k)} + \|Z\|^2_{M^2(\beta,\mathbf{R}^{k\times d})}.$$

Define $A = \Omega \times [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d}$. For each $t \in [0,T]$, we assume that we are given two functions:

$f : A \times C^2_G(\beta, a, [t, T+K], \mathbf{R}^k) \to L^2(\beta, G_t, \mathbf{R}^k)$,

$g : A \times C^2_G(\beta, a, [t, T+K], \mathbf{R}^k) \to L^2(\beta, G_t, \mathbf{R}^{k\times\ell})$.

For notational simplicity we assume: $h \in \{f, g\}$, $h(r, 0) = h(r, 0, 0, 0, 0)$ and for all $x, y \in \mathbf{R}^k$ we denote by $|x|$ the Euclidean norm of x and by $\langle x, y\rangle$ denote the Euclidean inner product.

Our ABDSDEs of interest are

$$\begin{cases} Y_t = \xi_T + \displaystyle\int_t^T f\left(s, \Theta_s, \Theta_s^{\delta,\zeta}\right) ds + \int_t^T g\left(s, \Theta_s, \Theta_s^{\delta,\zeta}\right) dB_s - \int_t^T Z_s dW_s, \quad 0 \le t \le T, \\ Y_t = \xi_t, \quad Z_t = \eta_t, \qquad T \le t \le T+K, \end{cases} \tag{1}$$

where K is a positive constant, $\Theta_s = (Y_s, Z_s)$, $\Theta_s^{\delta,\zeta} = (Y_{s+\delta(s)}, Z_{s+\zeta(s)})$ and $\delta, \zeta : [0,T] \to \mathbf{R}_+$ are continuous functions satisfying:

(**A1**): $t + \delta(t) \le T + K, \quad t + \zeta(t) \le T + K$;

(**A2**): There exists $M \ge 0$ such that for $0 \le t \le T$ and non negative integrable function h,

$\displaystyle\int_t^T h(s + \phi(s)) ds \le M \int_t^{T+K} h(s) ds, \quad \phi \in \{\delta, \zeta\}.$

Definition 2.1. A pair of processes (Y, Z) is called a solution to ABDSDEs (1) if $(Y, Z) \in B^2_G(\beta, a, [0, T+K], \mathbf{R}^k)$ and satisfies (1).

3. Anticipated BDSDEs

3.1. **Assumptions**

In the following, we assume that f and g satisfy assumptions (**H1**), namely there exists constants $0 < \alpha_1 < 1$, and $0 < \alpha_2 < 1/M$ satisfying $0 < \alpha_1 + \alpha_2 M < 1$, and three nonnegative processes $\{\mu(t)\}_{0\le t\le T}$, $\{\nu(t)\}_{0\le t\le T}$ and $\{\vartheta(t)\}_{0\le t\le T}$ such that:

(**H1.1**) for any $t \in [0,T]$ $\mu(t)$, $\nu(t)$ and $\vartheta(t)$ are $\mathcal{F}_t$ –measurable;

(**H1.2**) for all $t \in [0,T]$,

$r, r' \in [t, T+K]$, $(t,y,z,\theta(r),\varphi(r))$, $(t,y',z',\theta'(r'),\varphi'(r')) \in A \times C_G^2(\beta,a,[t,T+K],\mathbf{R}^k)$,

$$\begin{aligned}|f(t,y,z,\theta(r),\varphi(r)) - f(t,y',z',\theta'(r'),\varphi'(r'^2 &\le \mu(t)(|y-y'|^2 + \mathbf{E}^{\mathcal{F}t}[|\theta(r) - \theta'(r'^2]) \\ &\quad + \nu(t)(|z-z'|^2 + \mathbf{E}^{\mathcal{F}t}[|\varphi(r) - \varphi'(r'^2]), \\ |g(t,y,z,\theta(r),\varphi(r)) - g(t,y',z',\theta'(r'),\varphi'(r'^2 &\le \vartheta(t)(|y-y'|^2 + \mathbf{E}^{\mathcal{F}t}[|\theta(r) - \theta'(r'^2]) \\ &\quad + \alpha_1|z-z'|^2 + \alpha_2\mathbf{E}^{\mathcal{F}t}[|\varphi(r) - \varphi'(r'^2];\end{aligned}$$

(**H1.3**) for all $t \in [0,T+K]$, $a(t) = \mu(t) + \nu(t) + \vartheta(t) \ge 1$;

(**H1.4**) $\mathbf{E}\left[\int_0^T e^{\beta A(s)}|f(s,0)|^2 ds + \int_0^T e^{\beta A(s)}|g(s,0)|^2 ds\right] < +\infty.$

3.2. **Existence and uniqueness of solution**

To solve our equations, we examine first the case, where the coefficients do not depend on the variables. Namely, we consider the stochastic equation

$$Y_t = \xi_T + \int_t^T f(s)ds + \int_t^T g(s)dB_S - \int_t^T Z_S dW_S, \quad 0 \le t \le T, \tag{2}$$

where $\xi_T \in L^2(\beta, G_T, \mathbf{R}^k)$, and $(f,g) \in M^2_{[0,T]}(\beta, G, \mathbf{R}^k) \times M^2_{[0,T]}(\beta, G, \mathbf{R}^{k\times\ell})$ satisfy

(**H1.4**)$'$: $\mathbf{E}\left[\int_0^T e^{\beta A(s)}|f(s)|^2 ds + \int_0^T e^{\beta A(s)}|g(s)|^2 ds\right] < +\infty.$

Theorem 3.1. *For β sufficiently large and $\xi_T \in L^2(\beta, G_T, R^k)$, there exists a unique solution, $(Y,Z) \in B_G^2(\beta,a,[0,T],\mathbf{R}^k)$, of (2).*

Proof. The proof needs to be divided into several steps.

Step 1. Let us prove that $(Y,Z) \in C_G^2(0,1,[0,T],\mathbf{R}^k)$. First, we show that

$$\mathbf{E}\left[\left|\xi_T + \int_0^T f(s)ds + \int_0^T g(s)dB_S\right|^2\right] < +\infty. \tag{3}$$

We know that

$$\mathbf{E}\left[\left|\xi_T + \int_0^T f(s)ds + \int_0^T g(s)dB_s\right|^2\right] \le 3\mathbf{E}\left[|\xi_T|^2 + \left|\int_0^T f(s)ds\right|^2 + \int_0^T |g(s)|^2 ds\right].$$

Since $0 = A(0) \le A(t)$, for all $t \in [0,T]$, and for any $\beta > 0$, we obtain

$$\mathbf{E}\left[|\xi_T|^2+\int_0^T|g(s)|^2ds\right]\le\mathbf{E}\left[e^{\beta A(T)}|\xi_T|^2+\int_0^T e^{\beta A(s)}|g(s)|^2ds\right]<+\infty. \tag{4}$$

In view of Cauchy-Schwarz inequality and by integration, we have

$$\mathbf{E}\left[\left|\int_0^T f(s)ds\right|^2\right]=\mathbf{E}\left[\left|\int_0^T\sqrt{\frac{a(s)}{e^{\beta A(s)}}}\left(\sqrt{\frac{e^{\beta A(s)}}{a(s)}}f(s)\right)ds\right|^2\right]$$

$$\le\mathbf{E}\left[\left(\int_0^T a(s)e^{-\beta A(s)}ds\right)\left(\int_0^T\frac{e^{\beta A(s)}}{a(s)}|f(s)|^2ds\right)\right]$$

$$\le\mathbf{E}\left[\frac{1}{\beta}\int_0^T\frac{e^{\beta A(s)}}{a(s)}|f(s)|^2ds\right]. \tag{5}$$

Applying (**H1**.**3**), with (5), we deduce that

$$\mathbf{E}\left[\left|\int_0^T f(s)ds\right|^2\right]\le\mathbf{E}\left[\frac{1}{\beta}\int_0^T e^{\beta A(s)}|f(s)|^2ds\right]<+\infty. \tag{6}$$

Furthermore, by utilizing (4) and (6) we obtain (3).

Now, we define the filtration $\{H_t:\ t\in[0,T+K]\}$ by

$$H_t=\mathcal{F}^W_{0,t}\vee\mathcal{F}^B_{0,T+K}$$

and a H_t-square integrable martingale

$$M_t=\mathbf{E}^{H_t}\left[\xi_T+\int_0^T f(s)ds+\int_0^T g(s)dB_s\right],\quad t\in[0,T].$$

So, as in [6], we prove that (2) has a unique solution $(Y,Z)\in C^2_G(0,1,[0,T],\mathbf{R}^k)$.

Step 2. Let us prove that $(Y,Z)\in C^2_G(\beta,a,[0,T],\mathbf{R}^k)$.
Itô's formula applied to (2), for $t\in[0,T]$, yields

$$\begin{aligned}e^{\beta A(t)}|Y_t|^2+\beta\int_t^T e^{\beta A(s)}a(s)|Y_s|^2ds+\int_t^T e^{\beta A(s)}|Z_s|^2ds\\
=e^{\beta A(T)}|\xi_T|^2+2\int_t^T e^{\beta A(s)}\langle Y_s,f(s)\rangle ds-2\int_t^T e^{\beta A(s)}\langle Y_s,Z_sdW_s\rangle\\
+2\int_t^T e^{\beta A(s)}\langle Y_s,g(s)dB_s\rangle+\int_t^T e^{\beta A(s)}|g(s)|^2ds.\end{aligned} \tag{7}$$

Then, noting that

$$2\langle Y_s,f(s)\rangle\ \le\ \frac{\beta}{2}a(s)|Y_s|^2+\frac{2}{\beta}\frac{|f(s)|^2}{a(s)}\ \le\ \frac{\beta}{2}a(s)|Y_s|^2+\frac{2}{\beta}|f(s)|^2,$$

and taking expectations, we obtain

$$\begin{aligned}\mathbf{E}\frac{\beta}{2}\int_t^T e^{\beta A(s)}a(s)|Y_s|^2ds+\int_t^T e^{\beta A(s)}|Z_s|^2ds\\
\le\mathbf{E}\Big[e^{\beta A(T)}|\xi_T|^2+\frac{2}{\beta}\int_t^T e^{\beta A(s)}|f(s)|^2ds\\
+\int_t^T e^{\beta A(s)}|g(s)|^2ds\Big],\end{aligned} \tag{8}$$

which, in view of $(\mathbf{H1.4})'$ implies that $(Y,Z) \in C^2_G(\beta,a,[0,T],\mathbf{R}^k)$.

Step 3. Let us prove that $(Y,Z) \in B^2_G(\beta,a,[0,T],\mathbf{R}^k)$. From (7), we state, by Burkholder-Davis-Gundy's inequality, that for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right) &\le \mathbf{E}\left\{e^{\beta A(T)}|\xi_T|^2 + \frac{2}{\beta}\int_t^T e^{\beta A(s)}|f(s)|^2 ds + \int_t^T e^{\beta A(s)}|g(s)|^2 ds\right] \\
&\quad + 2\mathbf{E}\sup_{t\le r\le T}\left|\int_r^T e^{\beta A(s)}\langle Y_s, Z_s dW_s\rangle\right| + 2\mathbf{E}\sup_{t\le r\le T}\left|\int_r^T e^{\beta A(s)}\langle Y_s, g(s)dB_s\rangle\right| \\
&\le \mathbf{E}\left[e^{\beta A(T)}|\xi_T|^2 + \tfrac{2}{\beta}\int_t^T e^{\beta A(s)}|f(s)|^2 ds + \int_t^T e^{\beta A(s)}|g(s)|^2 ds\right] \\
&\quad + 2\varepsilon\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right) + \tfrac{1}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}|Z_s|^2 ds + \tfrac{1}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}|g(s)|^2 ds.
\end{aligned}$$

Hence, for $\varepsilon < \frac{1}{2}$, there exists $K(\varepsilon) > 0$ such that for $\beta > 1$, we have

$$\begin{aligned}
\mathbf{E}\left(\sup_{t\le r\le T} e^{\beta A(r)}|Y_r|^2\right) &\le K(\varepsilon)\mathbf{E}\left[e^{\beta A(T)}|\xi_T|^2 + \int_0^T e^{\beta A(s)}|f(s)|^2 ds\right. \\
&\qquad \left. + \int_0^T e^{\beta A(s)}|g(s)|^2 ds\right] < +\infty.
\end{aligned}$$

Therefore, the desired result is obtained. ■

Our strategy in the proof of the existence of solutions to Equation (1) is to use the Picard approximate sequence. To this end, we consider the sequence $(\Theta^n)_{n\ge 0} = (Y^n, Z^n)_{n\ge 0}$ given by

$$\begin{cases}
Y^0_t = 0, \quad 0 \le t \le T+K, \\
Y^{n+1}_t = \xi_T + \int_t^T f\left(s,\Theta^n_s,\Theta^{n,\delta,\zeta}_s\right)ds + \int_t^T g\left(s,\Theta^n_s,\Theta^{n,\delta,\zeta}_s\right)dB_s - \int_t^T Z^{n+1}_s dW_s, \\
0 \le t \le T, \\
Y^{n+1}_t = \xi_t, \quad Z^{n+1}_t = \eta_t, \qquad T \le t \le T+K.
\end{cases} \tag{9}$$

Denote $\overline{Y}^{n,m}_s = Y^n_s - Y^m_s$, $\overline{Z}^{n,m}_s = Z^n_s - Z^m_s$ and for a function $h \in \{f,g\}$, $\Delta h^{(n,m)}(s) = h\left(s,\Theta^n_s,\Theta^{n,\delta,\zeta}_s\right) - h\left(s,\Theta^m_s,\Theta^{m,\delta,\zeta}_s\right)$, for $n,m > 0$.
Then, it is obvious that $(\overline{Y}^{n+1,m+1},\overline{Z}^{n+1,m+1})$ solves the following ABDSDE

$$\begin{cases}
\overline{Y}^{n+1,m+1}_t = \int_t^T \Delta f^{(n,m)}(s)ds + \int_t^T \Delta g^{(n,m)}(s)dB_s - \int_t^T \overline{Z}^{n+1,m+1}_s dW_s, \qquad 0 \le t \le T, \\
\overline{Y}^{n+1,m+1}_t = 0, \quad \overline{Z}^{n+1,m+1}_t = 0, \qquad T \le t \le T+K.
\end{cases} \tag{10}$$

Next, we state the following result which will be useful in the sequel.

Proposition 3.1. A*ssume that (A1), (A2) and (H1) are true. There exists a constant* $K(\varepsilon) > 0$, *depending only on* ε *(with* $0 < \varepsilon < 1$*) such that for* $n,m \ge 1$ *we have*

$$\mathbf{E}\left(\sup_{t\leq r\leq T} e^{\beta A(r)}|\bar{Y}_r^{n+1,m+1}|^2\right) \leq K(\varepsilon)\mathbf{E}\Big[\int_0^T e^{\beta A(s)}a(s)|\bar{Y}_s^{n,m}|^2 ds + \int_0^T e^{\beta A(s)}|\bar{Z}_s^{n,m}|^2 ds\Big]. \tag{11}$$

Proof. Itô's formula applied to (10), for $t \in [0,T]$, yields

$$\begin{aligned} &e^{\beta A(t)}|\bar{Y}_t^{n+1,m+1}|^2 + \beta\int_t^T e^{\beta A(s)}a(s)|\bar{Y}_s^{n+1,m+1}|^2 ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s^{n+1,m+1}|^2 ds \\ &\quad = 2\int_t^T e^{\beta A(s)}\langle \bar{Y}_s^{n+1,m+1}, \Delta f^{(n,m)}(s)\rangle ds - 2\int_t^T e^{\beta A(s)}\langle \bar{Y}_s^{n+1,m+1}, \bar{Z}_s^{n+1,m+1}dW_s\rangle \\ &\quad + 2\int_t^T e^{\beta A(s)}\langle \bar{Y}_s^{n+1,m+1}, \Delta g^{(n,m)}(s)dB_s\rangle + \int_t^T e^{\beta A(s)}|\Delta g^{(n,m)}(s)|^2 ds. \end{aligned} \tag{12}$$

Using the fact that $\sqrt{b+d} \leq \sqrt{b} + \sqrt{d}$ and $2bd \leq \frac{1}{\varepsilon}b^2 + \varepsilon d^2$ (where $b, d, \varepsilon > 0$), we deduce from assumptions (**A1**), (**A2**), (**H1.1**), (**H1.2**) and (**H1.3**) that

$$\begin{aligned} &2\mathbf{E}\int_t^T e^{\beta A(s)}\langle \bar{Y}_s^{n+1,m+1}, \Delta f^{(n,m)}(s)\rangle ds \\ &\leq 2\mathbf{E}\int_t^T e^{\beta A(s)}|\bar{Y}_s^{n+1,m+1}|\sqrt{\mu(s)(|\bar{Y}_s^{n,m}|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Y}_{s+\delta(s)}^{n,m}|^2])}\, ds \\ &+ 2\mathbf{E}\int_t^T e^{\beta A(s)}|\bar{Y}_s^{n+1,m+1}|\sqrt{\nu(s)(|\bar{Z}_s^{n,m}|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Z}_{s+\zeta(s)}^{n,m}|^2])}\, ds \\ &\leq \varepsilon\mathbf{E}\int_t^T e^{\beta A(s)}[\mu(s)(|\bar{Y}_s^{n,m}|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Y}_{s+\delta(s)}^{n,m}|^2]) + (|\bar{Z}_s^{n,m}|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Z}_{s+\zeta(s)}^{n,m}|^2])]ds \\ &+ \frac{1}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}(1+\nu(s))|\bar{Y}_s^{n+1,m+1}|^2 ds \\ &\leq \varepsilon(1+M)\mathbf{E}\Big[\int_t^T e^{\beta A(s)}a(s)|\bar{Y}_s^{n,m}|^2 ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s^{n,m}|^2 ds\Big] \\ &+ \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}a(s)|\bar{Y}_s^{n+1,m+1}|^2 ds. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathbf{E}\int_t^T e^{\beta A(s)}|\Delta g^{(n,m)}(s)|^2 ds &\leq \mathbf{E}\int_t^T e^{\beta A(s)}\vartheta(s)(|\bar{Y}_s^{n,m}|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Y}_{s+\delta(s)}^{n,m}|^2])ds \\ &+ \mathbf{E}\int_t^T e^{\beta A(s)}(\alpha_1|\bar{Z}_s^{n,m}|^2 + \alpha_2\mathbf{E}^{\mathcal{F}_s}[|\bar{Z}_{s+\zeta(s)}^{n,m}|^2])ds \\ &\leq \mathbf{E}\Big[(1+M)\int_t^T e^{\beta A(s)}a(s)|\bar{Y}_s^{n,m}|^2 ds + (\alpha_1 + \alpha_2 M)\int_t^T e^{\beta A(s)}|\bar{Z}_s^{n,m}|^2 ds\Big] \\ &\leq \mathbf{E}\Big[(1+M)\int_t^T e^{\beta A(s)}a(s)|\bar{Y}_s^{n,m}|^2 ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s^{n,m}|^2 ds\Big]. \end{aligned}$$

Putting pieces together, we obtain

$$\mathbf{E}\int_t^T e^{\beta A(s)}\langle \overline{Y}_s^{n+1,m+1}, \Delta f^{(n,m)}(s)\rangle ds + \int_t^T e^{\beta A(s)}|\Delta g^{(n,m)}(s)|^2 ds$$
$$\leq K_1(\varepsilon)\mathbf{E}\Big[\int_t^T e^{\beta A(s)}a(s)|\overline{Y}_s^{n,m}|^2 ds + \int_t^T e^{\beta A(s)}|\overline{Z}_s^{n,m}|^2 ds\Big]$$
$$+ \frac{2}{\varepsilon}\mathbf{E}\Big[\int_t^T e^{\beta A(s)}a(s)|\overline{Y}_s^{n+1,m+1}|^2 ds\Big], \qquad (13)$$

where $K_1(\varepsilon) = (\varepsilon+1)(1+M)$. Next, choosing $\beta \geq \frac{2}{\varepsilon}+1$, from (12), leads to

$$\mathbf{E}\int_t^T e^{\beta A(s)}a(s)|\overline{Y}_s^{n+1,m+1}|^2 ds + \int_t^T e^{\beta A(s)}|\overline{Z}_s^{n+1,m+1}|^2 ds$$
$$\leq K_1(\varepsilon)\mathbf{E}\Big[\int_t^T e^{\beta A(s)}a(s)|\overline{Y}_s^{n,m}|^2 ds + \int_t^T e^{\beta A(s)}|\overline{Z}_s^{n,m}|^2 ds\Big]. \qquad (14)$$

Moreover, from (12), we have

$$\mathbf{E}\Big[\sup_{t\leq r\leq T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\Big] \leq 2\mathbf{E}\sup_{t\leq r\leq T}\Big|\int_r^T e^{\beta A(s)}\langle \overline{Y}_s^{n+1,m+1}, \Delta f^{(n,m)}(s)\rangle ds\Big|$$
$$+ 2\mathbf{E}\sup_{t\leq r\leq T}\Big|\int_r^T e^{\beta A(s)}\langle \overline{Y}_s^{n+1,m+1}, \Delta g^{(n,m)}(s)dB_s\rangle\Big|$$
$$+ 2\mathbf{E}\sup_{t\leq r\leq T}\Big|\int_r^T e^{\beta A(s)}\langle \overline{Y}_s^{n+1,m+1}, \overline{Z}_s^{n+1,m+1}dW_s\rangle\Big|$$
$$+ \mathbf{E}\int_t^T e^{\beta A(s)}|\Delta g^{(n,m)}(s)|^2 ds. \qquad (15)$$

By Burkholder-Davis-Gundy's inequality, there exists $\varepsilon > 0$ such that

$$2\mathbf{E}\sup_{t\leq r\leq T}\Big|\int_r^T e^{\beta A(s)}\langle \overline{Y}_s^{n+1,m+1}, \Delta g^{(n,m)}(s)dB_s\rangle\Big| \leq \frac{\varepsilon}{2}\mathbf{E}\Big[\sup_{t\leq r\leq T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\Big]$$
$$+ \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}|\Delta g^{(n,m)}(s)|^2 ds, \qquad (16)$$

and

$$2\mathbf{E}\sup_{t\leq r\leq T}\Big|\int_r^T e^{\beta A(s)}\langle \overline{Y}_s^{n+1,m+1}, \overline{Z}_s^{n+1,m+1}dW_s\rangle\Big| \leq \frac{\varepsilon}{2}\mathbf{E}\Big[\sup_{t\leq r\leq T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\Big]$$
$$+ \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}|\overline{Z}_s^{n+1,m+1}|^2 ds. \qquad (17)$$

By gathering (16) and (17) from (15) we deduce that

$$\mathbf{E}\left[\sup_{t\le r\le T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\right] \le 2\mathbf{E}\sup_{t\le r\le T}\left|\int_r^T e^{\beta A(s)}\langle \overline{Y}_s^{n+1,m+1}, \Delta f^{(n,m)}(s)\rangle ds\right|$$
$$+\frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}|\Delta g^{(n,m)}(s)|^2 ds + \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta A(s)}|\overline{Z}_s^{n+1,m+1}|^2 ds$$
$$+\varepsilon\mathbf{E}\left[\sup_{t\le r\le T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\right]. \quad (18)$$

Furthermore, by exploiting (13) we obtain

$$\mathbf{E}\left[\sup_{t\le r\le T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\right] \le \frac{\varepsilon^2+2}{\varepsilon(\varepsilon+1)}K_1(\varepsilon)\mathbf{E}\left[\int_t^T e^{\beta A(s)}a(s)|\overline{Y}_s^{n,m}|^2 ds\right.$$
$$\left.+\int_t^T e^{\beta A(s)}|\overline{Z}_s^{n,m}|^2 ds\right]$$
$$+\frac{2}{\varepsilon}\mathbf{E}\left[\int_t^T e^{\beta A(s)}a(s)|\overline{Y}_s^{n+1,m+1}|^2 ds + \int_t^T e^{\beta A(s)}|\overline{Z}_s^{n+1,m+1}|^2 ds\right]$$
$$+\varepsilon\mathbf{E}\left[\sup_{t\le r\le T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\right]. \quad (19)$$

Using (14) with (19) leads to

$$\mathbf{E}\left[\sup_{t\le r\le T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\right] \le K_2(\varepsilon)\mathbf{E}\left[\int_t^T e^{\beta A(s)}a(s)|\overline{Y}_s^{n,m}|^2 ds\right.$$
$$\left.+\int_t^T e^{\beta A(s)}|\overline{Z}_s^{n,m}|^2 ds\right]$$
$$+\varepsilon\mathbf{E}\left[\sup_{t\le r\le T} e^{\beta A(r)}|\overline{Y}_r^{n+1,m+1}|^2\right], \quad (20)$$

where $K_2(\varepsilon) = \frac{(\varepsilon+2)^2}{\varepsilon(\varepsilon+1)}K_1(\varepsilon)$.

Hence, from (20), for $\varepsilon < 1$ and $K(\varepsilon) = \frac{K_2(\varepsilon)}{1-\varepsilon}$, we obtain (11). Therefore, the desired result is reached. ■

The following theorem is the main result of this section. In this theorem, with the help of proposition 3.1, we can now prove our existence and uniqueness theorem.

Theorem 3.2. *Assume that (A1), (A2) and (H1) are true. Then for β sufficiently large and $(\xi,\eta) \in B_G^2(\beta,a,[T,T+K],\mathbf{R}^k)$, the ABDSDE (1) has a unique solution $(Y,Z) \in B_G^2(\beta,a,[0,T+K],\mathbf{R}^k)$.*

Proof. **i**) **Existence**:
Let us invoke the following equation

$$\begin{cases} \bar{Y}_t^{n+1} = \int_t^T \Delta f^{(n)}(s)ds + \int_t^T \Delta g^{(n)}(s)dB_s - \int_t^T \bar{Z}_s^{n+1} dW_s, & 0 \le t \le T, \\ \bar{Y}_t^{n+1} = 0, \quad \bar{Z}_t^{n+1} = 0, & T \le t \le T+K, \end{cases} \tag{21}$$

where $\bar{Y}_s^{n+1} = Y_s^{n+1} - Y_s^n$, $\bar{Z}_s^{n+1} = Z_s^{n+1} - Z_s^n$, and for a function $h \in \{f, g\}$,

$\Delta h^{(n)}(s) = h\left(s, \Theta_s^n, \Theta_s^{n,\delta,\zeta}\right) - h\left(s, \Theta_s^{n-1}, \Theta_s^{n-1,\delta,\zeta}\right).$

Itô's formula applied to (21), for $t \in [0,T]$, yields

$$\mathbf{E}\left[e^{A(t)}|\bar{Y}_t^{n+1}|^2 + \int_t^T e^{A(s)}a(s)|\bar{Y}_s^{n+1}|^2 ds + \int_t^T e^{A(s)}|\bar{Z}_s^{n+1}|^2 ds \right]$$
$$= \mathbf{E}\left[2\int_t^T e^{A(s)}\langle \bar{Y}_s^{n+1}, \Delta f^{(n)}(s)\rangle ds + \int_t^T e^{A(s)}|\Delta g^{(n)}(s)|^2 ds \right]. \tag{22}$$

By the same computations as before, we have

$$\mathbf{E}\left[(-\frac{2}{\varepsilon})\int_t^T e^{A(s)}a(s)|\bar{Y}_s^{n+1}|^2 ds + \int_t^T e^{A(s)}|\bar{Z}_s^{n+1}|^2 ds \right]$$
$$\le [\varepsilon + \alpha_1 + M(\varepsilon + \alpha_2)]\mathbf{E}\left[\frac{(\varepsilon+1)(1+M)}{\varepsilon + \alpha_1 + M(\varepsilon+\alpha_2)} \int_t^T e^{A(s)}a(s)|\bar{Y}_s^n|^2 ds + \int_t^T e^{A(s)}|\bar{Z}_s^n|^2 ds \right].$$

Hence, if we choose $\varepsilon = \varepsilon_0$ satisfying $C_0 = \varepsilon_0 + \alpha_1 + M(\varepsilon_0 + \alpha_1) < 1$, let $\beta = \frac{2}{\varepsilon_0} + \frac{(1+M)(1+\varepsilon_0)}{C_0}$ and denote $\bar{C}_0 = \frac{(1+M)(1+\varepsilon_0)}{C_0}$, to write

$$\mathbf{E}\left[\bar{C}_0 \int_t^T e^{A(s)}a(s)|\bar{Y}_s^{n+1}|^2 ds + \int_t^T e^{A(s)}|\bar{Z}_s^{n+1}|^2 ds \right]$$
$$\le C_0 \mathbf{E}\left[\bar{C}_0 \int_t^T e^{A(s)}a(s)|\bar{Y}_s^n|^2 ds + \int_t^T e^{A(s)}|\bar{Z}_s^n|^2 ds \right],$$

which implies

$$\mathbf{E}\bar{C}_0 \int_t^{T+K} e^{A(s)}a(s)|\bar{Y}_s^{n+1}|^2 ds + \int_t^{T+K} e^{A(s)}|\bar{Z}_s^{n+1}|^2 ds$$
$$\le C_0 \mathbf{E}\left[\bar{C}_0 \int_t^{T+K} e^{A(s)}a(s)|\bar{Y}_s^n|^2 ds + \int_t^{T+K} e^{A(s)}|\bar{Z}_s^n|^2 ds \right].$$

Here we are able to deduce that $(Y^n, Z^n)_{n\ge 1}$ is a Cauchy sequence in $C_G^2(, a, [0, T+K], \mathbf{R}^k)$.

It remains necessary to show that $(Y^n, Z^n)_{n\ge 1}$ is a Cauchy sequence in $B_G^2(, a, [0, T+K], \mathbf{R}^k)$. Then, by proposition 3.1 we have

$$\mathbf{E}\left[\sup_{0\le t\le T+K} e^{A(t)}|\bar{Y}_t^{n+1,m+1}|^2 \right] \le K(\varepsilon)\mathbf{E}\left[\int_0^{T+K} e^{A(s)}a(s)|\bar{Y}_s^{n,m}|^2 ds + \int_0^{T+K} e^{A(s)}|\bar{Z}_s^{n,m}|^2 ds \right].$$

Since $(Y^n, Z^n)_{n\ge 1}$ is a Cauchy sequence in $C_G^2(\beta, a, [0, T+K], \mathbf{R}^k)$, we deduce that $(Y^n)_{n\ge 1}$ is a Cauchy sequence in $S^2_{[0,T+K]}(\beta, G, \mathbf{R}^k)$. Hence, $(Y^n, Z^n)_{n\ge 1}$ is a Cauchy sequence in $B_G^2(\beta, a, [0, T+K], \mathbf{R}^k)$. Then there exists (Y, Z) in $B_G^2(\beta, a, [0, T+K], \mathbf{R}^k)$ being a limit of $(Y^n, Z^n)_{n\ge 1}$.

Now, let us show that (Y, Z) is a solution to ABDSDE (1). Since $(Y^n, Z^n)_{n\ge 1}$ converges in $B_G^2(, a, [0, T+K], \mathbf{R}^k)$ to (Y, Z), we have

$$\lim_{n\to+\infty} \mathbf{E}\left[\sup_{0\le t\le T} e^{A(t)}|Y_t^n - Y_t|^2 + \int_0^T e^{A(s)}a(s)|Y_s^n - Y_s|^2 ds + \int_0^T e^{A(s)}|Z_s^n - Z_s|^2 ds\right] = 0$$

and $\quad Y_t^n = \xi_t,\ Z_t^n = \eta_t \qquad T \le t \le T+K.$

Hence, let us define a function $h \in \{f,g\}$, $\ h^{(n)}(s) = h\left(s, Y_s^n,\ Z_s^n,\ Y_{s+\delta(s)}^n,\ Z_{s+\zeta(s)}^n\right)$ and $h(s) = h\left(s,\ Y_s,\ Z_s,\ Y_{s+\delta(s)},\ Z_{s+\zeta(s)}\right)$, to write

$$\mathbf{E}\left|\int_t^T \left(f^{(n)}(s) - f(s)\right)ds\right|^2 = \mathbf{E}\left[\left|\int_t^T \sqrt{\frac{a(s)}{e^{A(s)}}}\left(\sqrt{\frac{e^{A(s)}}{a(s)}}\left(f^{(n)}(s) - f(s)\right)\right)ds\right|^2\right]$$

$$\le \mathbf{E}\left[\left(\int_0^T a(s)e^{-A(s)}ds\right)\left(\int_0^T \frac{e^{A(s)}}{a(s)}|f^{(n)}(s) - f(s)|^2 ds\right)\right]$$

$$\le \frac{1}{\beta}(1+M)\mathbf{E}\left[\int_0^T \frac{e^{A(s)}}{a(s)}\mu(s)|Y_S^n - Y_S|^2 ds + \int_0^T \frac{e^{A(s)}}{a(s)}\nu(s)|Z_S^n - Z_S|^2 ds\right]$$

$$\le \frac{1}{\beta}(1+M)\mathbf{E}\left[\int_0^T e^{A(s)}|Y_S^n - Y_S|^2 ds + \int_0^T e^{A(s)}|Z_S^n - Z_S|^2 ds\right]$$

$$\le \frac{1}{\beta}(1+M)\mathbf{E}\left[\int_0^T e^{A(s)}a(s)|Y_S^n - Y_S|^2 ds + \int_0^T e^{A(s)}|Z_S^n - Z_S|^2 ds\right]. \tag{23}$$

Letting $n \to +\infty$ in (23), we deduce that

$$\mathbf{E}\left[\left|\int_t^T \left(f\left(s, Y_S^n, Z_S^n, Y_{s+\delta(s)}^n, Z_{s+\zeta(s)}^n\right) - f\left(s, Y_S, Z_S, Y_{S+\delta(s)}, Z_{S+\zeta(s)}\right)\right)ds\right|^2\right] \to 0. \tag{24}$$

On the other hand, we have

$$\mathbf{E}\left[\left|\int_t^T \left(g^{(n)}(s) - g(s)\right)dB_S\right|^2\right] \le \mathbf{E}\left[\int_t^T \left|g^{(n)}(s) - g(s)\right|^2 ds\right]$$

$$\le (1+M)\mathbf{E}\left[\int_0^T e^{A(s)}a(s)|Y_S^n - Y_S|^2 ds + \int_0^T e^{A(s)}|Z_S^n - Z_S|^2 ds\right] \to 0, \tag{25}$$

as $n \to +\infty$, and

$$\mathbf{E}\left[\left|\int_t^T (Z_S^n - Z_S)dW_S\right|^2\right] \le \mathbf{E}\left[\int_0^T e^{A(s)}|Z_S^n - Z_S|^2 ds\right] \to 0, \text{ as } n \to +\infty. \tag{26}$$

Furthermore, by utilizing (24), (25), (26) and passing to the limit in (9), we obtain

$$\begin{cases} Y_t = \xi_T + \int_t^T f\left(s, \Theta_s, \Theta_s^{\delta,\zeta}\right)ds + \int_t^T g\left(s, \Theta_s, \Theta_s^{\delta,\zeta}\right)dB_s - \int_t^T Z_s dW_s, & 0 \le t \le T, \\ Y_t = \xi_t, \quad Z_t = \eta_t, & T \le t \le T+K. \end{cases}$$

This shows that $(Y,Z) \in B_G^2(\beta, a, [0, T+K], \mathbf{R}^k)$ satisfy the ABDSDE (1).

ii) Uniqueness:

Let (Y,Z) and (Y',Z') be two solutions of the ABDSDE (1). Then let $\bar{Y}_s = Y_s - Y_s'$, $\bar{Z}_s = Z_s - Z_s'$ in a function $h \in \{f,g\}$,

$$\Delta h(s) = h\left(s,\ Y_s,\ Z_s,\ Y_{s+\delta(s)},\ Z_{s+\zeta(s)}\right) - h\left(s,\ Y_s',\ Z_s',\ Y_{s+\delta(s)}',\ Z_{s+\zeta(s)}'\right).$$

Here it is obvious that $(\bar{Y}, \bar{Z})$ is a solution in $B_G^2(\beta, a, [0, T+K], \mathbf{R}^k)$ to the following ABDSDE

$$\begin{cases} \bar{Y}_t = \int_t^T \Delta f(s)ds + \int_t^T \Delta g(s)dB_s - \int_t^T \bar{Z}_s dW_s, & 0 \le t \le T, \\ \bar{Y}_t = 0, \quad \bar{Z}_t = 0, & T \le t \le T+K. \end{cases} \tag{27}$$

Itô's formula applied to (27), for $t \in [0,T]$, yields

$$\mathbf{E}e^{\beta A(t)}|\bar{Y}_t|^2 + \int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s|^2 ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s|^2 ds$$
$$= \mathbf{E}\Big[2\int_t^T e^{\beta A(s)}\langle \bar{Y}_s, \Delta f(s)\rangle ds + \int_t^T e^{\beta A(s)}|\Delta g(s)|^2 ds \Big]. \tag{28}$$

Using the fact that $\sqrt{b+d} \le \sqrt{b} + \sqrt{d}$ and $2bd \le \frac{1}{\varepsilon}b^2 + \varepsilon d^2$ (where $b, d, \varepsilon > 0$), we deduce from assumptions (**A1**), (**A2**), (**H1.1**), (**H1.2**) and (**H1.3**) that

$$\begin{aligned} 2\mathbf{E}\Big[\int_t^T e^{\beta A(s)}\langle \bar{Y}_s, \Delta f(s)\rangle ds \Big] &\le 2\mathbf{E}\Big[\int_t^T e^{\beta A(s)}|\bar{Y}_s|\sqrt{\mu(s)(|\bar{Y}_s|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Y}_{s+\delta(s)}|^2])}\, ds \Big] \\ &+ 2\mathbf{E}\Big[\int_t^T e^{\beta A(s)}|\bar{Y}_s|\sqrt{\nu(s)(|\bar{Z}_s|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Z}_{s+\zeta(s)}|^2])}\, ds \Big] \\ &\le \mathbf{E}\Big[\int_t^T e^{\beta A(s)}\Big(\frac{\varepsilon}{2}\mu(s)|\bar{Y}_s|^2 + \frac{2}{\varepsilon}(|\bar{Y}_s|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Y}_{s+\delta(s)}|^2])\Big) ds \Big] \\ &+ \mathbf{E}\Big[\int_t^T e^{\beta A(s)}\Big(\frac{\varepsilon}{2}\nu(s)|\bar{Y}_s|^2 + \frac{2}{\varepsilon}(|\bar{Z}_s|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Z}_{s+\zeta(s)}|^2])\Big) ds \Big] \\ &\le \mathbf{E}\Big[\frac{\varepsilon}{2}\int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s|^2 ds + \frac{2}{\varepsilon}(1+M)\int_t^T e^{\beta A(s)}|\bar{Z}_s|^2 ds \Big] \\ &+ \frac{2}{\varepsilon}(1+M)\mathbf{E}\Big[\int_t^T e^{\beta A(s)}|\bar{Y}_s|^2 ds \Big], \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}\Big[\int_t^T e^{\beta A(s)}|\Delta g(s)|^2 ds \Big] &\le \mathbf{E}\Big[\int_t^T e^{\beta A(s)}\vartheta(s)(|\bar{Y}_s|^2 + \mathbf{E}^{\mathcal{F}_s}[|\bar{Y}_{s+\delta(s)}|^2]) ds \Big] \\ &+ \mathbf{E}\Big[\int_t^T e^{\beta A(s)}(\alpha_1|\bar{Z}_s|^2 + \alpha_2\mathbf{E}^{\mathcal{F}_s}[|\bar{Z}_{s+\zeta(s)}|^2]) ds \Big] \\ &\le \mathbf{E}\Big[(1+M)\int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s|^2 ds + (\alpha_1 + \alpha_2 M)\int_t^T e^{\beta A(s)}|\bar{Z}_s|^2 ds \Big]. \end{aligned}$$

Putting pieces together, we obtain

$$\begin{aligned} &\mathbf{E}\Big[e^{\beta A(t)}|\bar{Y}_t|^2 + \beta\int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s|^2 ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s|^2 ds \Big] \\ &\le \mathbf{E}\Big[(\frac{\varepsilon}{2} + 1 + M)\int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s|^2 ds + \Big(\alpha_1 + \alpha_2 M + \frac{2}{\varepsilon}(1+M)\Big)\int_t^T e^{\beta A(s)}|\bar{Z}_s|^2 ds \Big] \\ &+ \frac{2}{\varepsilon}(1+M)\mathbf{E}\Big[\int_t^T e^{\beta A(s)}|\bar{Y}_s|^2 ds \Big]. \end{aligned}$$

Hence if we choose $\varepsilon = \varepsilon_0$ satisfying $C_0 = \frac{2}{\varepsilon_0}(1+M) + \alpha_1 + \alpha_2 M < 1$, and $\beta \ge \bar{C}_0 + \frac{2}{\varepsilon_0} + 1 + M$, where $\bar{C}_0 = 1 - C_0$, we deduce that

$$\mathbf{E}\Big[e^{\beta A(t)}|\bar{Y}_t|^2\Big] + \bar{C}_0\mathbf{E}\Big[\int_t^T e^{\beta A(s)}a(s)|\bar{Y}_s|^2ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s|^2ds\Big] \leq \frac{2}{\varepsilon_0}(1+M)\mathbf{E}\Big[\int_t^T e^{\beta A(s)}|\bar{Y}_s|^2ds\Big]. \tag{29}$$

This leads to

$$\mathbf{E}\Big[e^{\beta A(t)}|\bar{Y}_t|^2\Big] \leq \frac{2}{\varepsilon_0}(1+M)\mathbf{E}\Big[\int_t^{T+K} e^{\beta A(s)}|\bar{Y}_s|^2ds\Big], \qquad 0 \leq t \leq T+K.$$

Hence by applying Gronwall's inequality, we arrive at

$$\mathbf{E}\Big[e^{\beta A(t)}|\bar{Y}_t|^2\Big] = 0, \qquad 0 \leq t \leq T+K.$$

Therefore, $\bar{Y}_t = 0$, a.s., $\forall t \in [0,T+K]$, and from inequality (29), we obtain $\bar{Z}_t = 0$, a.s., $\forall t \in [0,T+K]$. Here the proof completes. ■

4. Comparison Theorem

In this section, we shall formulate and prove a comparison theorem for some ABDSDEs with stochastic Lipschitz coefficients. From now on, we only consider the following 1-dimensional ABDSDEs (i.e. $k = 1$):

$$\begin{cases} 0 \leq t \leq T, \\ Y_t^i = \xi_T^i + \int_t^T f^i\Big(s,\Theta_s^i,Y_{s+\delta(s)}^i,Z_{s+\zeta^i(s)}^i\Big)ds + \int_t^T g\Big(s,\Theta_s^i,Y_{s+\delta(s)}^i,Z_{s+\zeta^i(s)}^i\Big)dB_s \\ -\int_t^T Z_s^i dW_s, \\ Y_t^i = \xi_t^i, \quad Z_t^i = \eta_t^i, \qquad T \leq t \leq T+K, \end{cases} \tag{30}$$

where $i = 1,2$, and $(\xi^i,\eta^i) \in B_G^2(\beta,a,[T,T+K],\mathbf{R})$, $(\delta(.),\zeta^i(.))$ satisfy (**A1**) and (**A2**), and (f^i,g) satisfy (**H1**). Then, by theorem 3.2, equation (30) has a unique solution.

Our objective is to obtain a comparison result for subsequent two equations. For this purpose, we first consider a simple case when the coefficients f^i and g do not depend on the future value of (Y^i,Z^i) via

$$Y_t^i = \xi_T^i + \int_t^T \tilde{f}^i\Big(s,Y_s^i,Z_s^i\Big)ds + \int_t^T \tilde{g}\Big(s,Y_s^i,Z_s^i\Big)dB_s - \int_t^T Z_s^i dW_s, \quad 0 \leq t \leq T, \tag{31}$$

where $i = 1,2$, when f^i and g do not depend on δ,ζ.

Now we assume that f and g satisfy assumptions (**B**). In particular

(**B.1**) Condition (**H1.1**) holds;

(**B.2**) for all $t \in [0,T]$ and $(y,z),(y',z') \in \mathbf{R}\times\mathbf{R}^d$,

$$|\tilde{f}(t,y,z) - \tilde{f}(t,y',z'^2 \leq \mu(t)|y-y'|^2 + \nu(t)|z-z'|^2$$
$$|\tilde{g}(t,y,z) - \tilde{g}(t,y',z'^2 \leq \vartheta(t)|y-y'|^2 + \alpha_1|z-z'|^2;$$

(**B.3**) for all $t \in [0,T]$, $a(t) = \mu(t) + \nu(t) + \vartheta(t) \geq 1$;

(**B.4**) $\mathbf{E}\Big[\int_0^T e^{\beta A(s)}|\tilde{f}(s,0,0)|^2ds + \int_0^T e^{\beta A(s)}|\tilde{g}(s,0,0)|^2ds\Big] < +\infty$.

The next theorem will be useful in the sequel.

Theorem 4.1. *Suppose that $\tilde{f}^1$, $\tilde{g}$ and $\tilde{f}^2$ satisfy (B) and $(\xi_T^1, \xi_T^2) \in L^2(\beta, G_T, \mathbf{R}) \times L^2(\beta, G_T, \mathrm{R})$. Let $(Y^i, Z^i) \in S^2_{[0,T]}(\beta, G, \mathbf{R}) \times M^2_{[0,T]}(\beta, G, \mathbf{R}^d)$ $(i = 1, 2)$ be the unique solutions to BDSDEs (31) respectively. If $\xi_T^1 \geq \xi_T^2$, a.s., and for any $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$,*

$$\tilde{f}^1(t,y,z) \geq \tilde{f}^2(t,y,z), \quad \text{a.s.},$$

then

$$Y_t^1 \geq Y_t^2, \text{ a.s.}, \quad \text{for all } t \in [0, T].$$

Proof. Let $\bar{Y}_s = Y_s^2 - Y_s^1$, $\bar{Z}_s = Z_s^2 - Z_s^1$, $\bar{\xi}_T = \xi_T^2 - \xi_T^1$, $\Delta f(s) = \tilde{f}^2(s, Y_s^2, Z_s^2) - \tilde{f}^1(s, Y_s^1, Z_s^1)$ and $\Delta g(s) = \tilde{g}(s, Y_s^2, Z_s^2) - \tilde{g}(s, Y_s^1, Z_s^1)$. Then consider the following equation

$$\begin{cases} \bar{Y}_t = \int_t^T \Delta f(s)ds + \int_t^T \Delta g(s)dB_s - \int_t^T \bar{Z}_s dW_s, \qquad 0 \leq t \leq T, \\ \bar{Y}_t = 0, \quad \bar{Z}_t = 0, \qquad T \leq t \leq T + K. \end{cases}$$

Applying Itô's formula to $e^{\beta A(t)}|\bar{Y}_t^+|^2$ and noting that $\xi_T^1 \geq \xi_T^2$, we have

$$\begin{aligned} &\mathbf{E} e^{\beta A(t)}|\bar{Y}_t^+|^2 + \beta \int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s^+|^2 ds + \int_t^T \Uparrow_{\{Y_s^2 \geq Y_s^1\}} e^{\beta A(s)}|\bar{Z}_s|^2 ds \\ &\qquad = \mathbf{E}\left[2\int_t^T e^{\beta A(s)} \bar{Y}_s^+ \Delta f(s) ds + \int_t^T \Uparrow_{\{Y_s^2 \geq Y_s^1\}} e^{\beta A(s)}|\Delta g(s)|^2 ds \right]. \end{aligned} \tag{32}$$

Now, for any $\varepsilon > 0$, we obtain from assumption (**B.2**)

$$\begin{aligned} \mathbf{E}\left[2\int_t^T e^{\beta A(s)} \bar{Y}_s^+ \Delta f(s) ds \right] &\leq 2\mathbf{E}\left[\int_t^T e^{\beta A(s)}|\bar{Y}_s^+|\left(\sqrt{\mu(s)|\bar{Y}_s^+|^2} + \sqrt{\nu(s)|\bar{Z}_s|^2} \right) ds \right] \\ &\leq \frac{\varepsilon}{2}\mathbf{E}\left[\int_t^T e^{\beta A(s)}|\bar{Y}_s^+|^2 ds + \int_t^T e^{\beta A(s)} \Uparrow_{\{Y_S^2 \geq Y_S^1\}} |\bar{Z}_s|^2 ds \right] \\ &\quad + \frac{2}{\varepsilon}\mathbf{E}\left[\int_t^T e^{\beta A(s)}(\mu(s) + \nu(s))|\bar{Y}_s^+|^2 ds \right] \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}\left[\int_t^T \Uparrow_{\{Y_S^2 \geq Y_S^1\}} e^{\beta A(s)}|\Delta g(s)|^2 ds \right] &\leq \mathbf{E}\left[\int_t^T e^{\beta A(s)} \vartheta(s)|\bar{Y}_s^+|^2 ds \right. \\ &\qquad \left. + \alpha_1 \int_t^T e^{\beta A(s)} \Uparrow_{\{Y_S^2 \geq Y_S^1\}} |\bar{Z}_s|^2 ds \right]. \end{aligned}$$

Consequently, performing the same computation as before, allows for

$$\begin{aligned} &\mathbf{E}\left[e^{\beta A(t)}|\bar{Y}_t^+|^2 + \beta\int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s^+|^2 ds + \int_t^T \Uparrow_{\{Y_S^2 \geq Y_S^1\}} e^{\beta A(s)}|\bar{Z}_s|^2 ds \right] \\ &\quad \leq \mathbf{E}\left[(\frac{2}{\varepsilon} + 1)\int_t^T e^{\beta A(s)} a(s)|\bar{Y}_s^+|^2 ds + (\frac{\varepsilon}{2} + \alpha_1)\int_t^T e^{\beta A(s)} \Uparrow_{\{Y_S^2 \geq Y_S^1\}} |\bar{Z}_s|^2 ds \right] \\ &\quad + \frac{\varepsilon}{2}\mathbf{E}\left[\int_t^T e^{\beta A(s)}|\bar{Y}_s^+|^2 ds \right]. \end{aligned}$$

Choosing $\varepsilon = \varepsilon_0$ such that $\frac{\varepsilon_0}{2} < 1 - \alpha_1$, and taking $> \frac{2}{\varepsilon_0} + 1$, we obtain

$$\mathbf{E}\Big[e^{\beta A(t)}|\bar{Y}_t^+|^2\Big] \le \frac{\varepsilon_0}{2}\int_t^T \mathbf{E}\Big[e^{\beta A(s)}|\bar{Y}_s^+|^2\Big]ds, \quad 0 \le t \le T.$$

Now we can use Gronwall's inequality to get $|\bar{Y}_t^+|^2 = 0,\ a.s., \quad 0 \le t \le T.$
This shows that

$$Y_t^1 \ge Y_t^2, \text{ a.s}, \quad 0 \le t \le T.$$

The proof is therefore complete. ■

Next, let us turn to the study of the comparison theorem for ABDSDEs (30). For this purpose, we assume in addition (**H2**) that
(**H2.1**) for all $t \in [0,T]$, $z \in \mathbf{R}^d$ and $\phi(s) \in L^2(\beta, G_s, \mathbf{R})$, $\varphi(s) \in L^2(\beta, G_s, \mathbf{R}^d)$, $s \in [t, T+K]$, $f^2(t,.,z,\phi(s),\varphi(s))$ is nondecreasing.
(**H2.2**) for all $t \in [0,T]$, $(y,z) \in \mathbf{R}\times\mathbf{R}^d$ and $\varphi(s) \in L^2(\beta, G_s, \mathbf{R}^d)$, $s \in [t, T+K]$, $f^2(t,y,z,.,\varphi(s))$ is nondecreasing.

Then, we are in position to prove the main result of this section.

Theorem 4.2. *Under (H1)and (H2), assume that*
(i) $\xi_t^1 \ge \xi_t^2$, *a.s.*, $T \le t \le T+K$;
(ii) $f^1(t,y,z,\phi(s),Z^1_{t+\zeta^1(t)}) \ge f^2(t,y,z,\phi(s),Z^2_{t+\zeta^2(t)})$, *a.s.* $t \in [0,T]$, $s \in [t,T+K]$, $y \in R$, $z \in R^d$.
Then $Y_t^1 \ge Y_t^2$, a.s., $0 \le t \le T+K$.

Proof. Let us introduce the following equation

$$\begin{cases} 0 \le t \le T, \\ Y_t^3 = \xi_T^2 + \int_t^T f^2(s,\Theta_s^3, Y^1_{s+\delta(s)}, Z^2_{s+\zeta^2(s)})ds + \int_t^T g(s,\Theta_s^3, Y^1_{s+\delta(s)}, Z^2_{s+\zeta^2(s)})dB_s - \int_t^T Z_s^3 dW_s, \\ Y_t^3 = \xi_t^2, \quad Z_t^3 = \eta_t^2, \qquad T \le t \le T+K. \end{cases}$$

From the proof of theorem 3.2, there exists a unique pair $\Theta^3 = (Y^3, Z^3) \in B_G^2(\beta, a, [0,T+K], \mathbf{R})$ that satisfies the above ABDSDE.
If we set $\tilde{f}^2(s,y,z) = f^2(s,y,z,Y^1_{s+\delta(s)}, Z^2_{s+\zeta^2(s)})$ and $\tilde{g}^2(s,y,z) = g(s,y,z,Y^1_{s+\delta(s)}, Z^2_{s+\zeta^2(s)})$, then this equation is equivalent to

$$Y_t^3 = \xi_T^2 + \int_t^T \tilde{f}^2(s,Y_s^3,Z_s^3)ds + \int_t^T \tilde{g}^2(s,Y_s^3,Z_s^3)dB_s - \int_t^T Z_s^3 dW_s, \quad 0 \le t \le T.$$

On the other hand, if we set $\tilde{f}^1(s,Y_s^1,Z_s^1) = f^1(s,Y_s^1,Z_s^1,Y^1_{s+\delta(s)},Z^1_{s+\zeta^1(s)})$ and $\tilde{g}^1(s,Y_s^1,Z_s^1) = g(s,Y_s^1,Z_s^1,Y^1_{s+\delta(s)},Z^1_{s+\zeta^1(s)})$, then (Y^1,Z^1) is also the unique solution of $Y_t^1 = \xi_T^1 + \int_t^T \tilde{f}^1(s,Y_s^1,Z_s^1)ds + \int_t^T \tilde{g}^1(s,Y_s^1,Z_s^1)dB_s - \int_t^T Z_s^1 dW_s, \quad 0 \le t \le T.$

From the assumptions (**i**) and (**ii**) it follows that $\xi_T^1 \ge \xi_T^2$, a.s., and $\tilde{f}^1(t,Y_t^1,Z_t^1) \ge \tilde{f}^2(t,Y_t^3,Z_t^3)$ for all $t \in [0,T]$. By theorem 4.1, we obtain $Y_t^1 \ge Y_t^3$ a.s., for all most $t \in [0,T]$. Thus, we have

$$Y_t^1 \ge Y_t^3, \text{ a.s.}, \quad 0 \le t \le T+K.$$

Then set

$$\begin{cases} 0 \le t \le T, \\ Y_t^4 = \xi_T^2 + \int_0^T f^2\Big(s, \Theta_s^3, Y_{s+\delta(s)}^3, Z_{s+\zeta^2(s)}^2\Big)ds + \int_0^T g\Big(s, \Theta_s^3, Y_{s+\delta(s)}^3, Z_{s+\zeta^2(s)}^2\Big)dB_s - \int_0^T Z_s^4 dW_s, \\ Y_t^4 = \xi_t^2, \quad Z_t^4 = \eta_t^2, \qquad T \le t \le T+K. \end{cases}$$

Since, for $t \in [0,T]$, $y \in \mathbf{R}$, $z \in \mathbf{R}^d$, f^2 satisfies (**H2**), and $Y_t^1 \ge Y_t^3$, by theorem 4.1, we know that

$Y_t^3 \ge Y_t^4$, a.s., $\quad 0 \le t \le T+K.$

Now, for $n = 4, 5, 6, \ldots$, we consider the following equation

$$\begin{cases} 0 \le t \le T, \\ Y_t^{n+1} = \xi_T^2 + \int_t^T f^2\Big(s, \Theta_s^n, Y_{s+\delta(s)}^n, Z_{s+\zeta^2(s)}^2\Big)ds + \int_t^T g\Big(s, \Theta_s^n, Y_{s+\delta(s)}^n, Z_{s+\zeta^2(s)}^2\Big)dB_s \\ -\int_t^T Z_s^{n+1} dW_s, \\ Y_t^{n+1} = \xi_t^2, \quad Z_t^{n+1} = \eta_t^2, \qquad T \le t \le T+K. \end{cases} \tag{33}$$

Thanks to theorem 4.1, by induction we have $Y_t^4 \ge Y_t^5 \ge \ldots \ge Y_t^{n-1} \ge Y_t^n$, a.s., $0 \le t \le T+K$. Therefore, for $n \ge 5$,

$Y_t^1 \ge Y_t^n$, a.s., $\quad 0 \le t \le T+K.$

From the proof of theorem 3.2, we know that (Y^n, Z^n) is a Cauchy sequences in $B_G^2(\beta, a, [0,T], \mathbf{R})$. Hence we may denote their limits by (Y, Z), and take limits in (33), to obtain that (Y, Z) satisfies the following ABDSDE

$$\begin{cases} 0 \le t \le T, \\ Y_t = \xi_T^2 + \int_t^T f^2\Big(s, \Theta_s, Y_{s+\delta(s)}, Z_{s+\zeta^2(s)}^2\Big)ds + \int_t^T g\Big(s, \Theta_s, Y_{s+\delta(s)}, Z_{s+\zeta^2(s)}^2\Big)dB_s - \int_t^T Z_s dW_s, \\ Y_t = \xi_t^2, \quad Z_t = \eta_t^2, \qquad T \le t \le T+K. \end{cases}$$

Then the uniqueness part of theorem 3.2 leads to

$Y_t^2 = Y_t$, a.s., $\quad 0 \le t \le T+K,$

which implies that $Y_t^4 \ge Y_t^5 \ge \ldots \ge Y_t^{n-1} \ge Y_t^n$, a.s., $0 \le t \le T+K$.

This leads to the required result
$Y_t^1 \ge Y_t^2$, a.s., $\quad 0 \le t \le T+K.$ ■

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