Journal of Numerical Mathematics and Stochastics, **2** (1) : 34-44, 2010 © ***JNM@S***
http://www.jnmas.org/jnmas2-3.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

Semi-Analytical Approach to the Shock Wave Equation

A. YILDIRIM[1], **S.T. MOHYUD – DIN**[2] **and M.E. BERBERLER**[1]

[1]Department of Mathematics, Ege University, 35100 Bornova-Ýzmir, Turkey; [2]HITEC University Taxila Cantt, Pakistan. E-mail: syedtauseefs@hotmail.com

Abstract. *In this paper we discuss an analytical solution for fully developed shock waves. The homotopy analysis method is applied to solve the shock wave equation for a flow of gases. Unlike various alternative numerical techniques, which are usually valid for a limited duration of time, t, the presented solution of this equation is valid for* $t \in (0, \infty)$ *. The reported numerical results reveal a reliability of the proposed algorithm.*

Key words : Homotopy Analysis Method, Shock Wave Equations, Soliton Solutions, Conservation Law.

AMS Subject Classifications : 65H20, 65N99, 65Z05

1. Introduction

The rapid development of nonlinear sciences has witnessed a number of new and reliable techniques including the homotopy analysis method (HAM) [15-19, 21-23] which was advanced in 1992 by Liao. The method has successfully been implemented on a wide range of nonlinear problems, see [1, 2, 5, 7, 12, 13, 20] and the references therein. The method yields a solution in terms of a convergent series with easily computable coefficients. An auxiliary parameter, $\hbar$, pertaining to this method provides for a simple way to adjust and control the convergence region of the solution series for large values of *t*.

Conservation laws are common features of various theories in continuum physics. These laws are supplemented by consistency relations which characterize the particular medium in question by relating values of the main field, u, to the flux, f , of this field.This is done under the assumption that these relations are smooth in their variation. Consequently the conservation laws lead to nonlinear hyperbolic partial differential equations. The simplest type of these is the first-order differential equation:

$$u_t(x,t) + f(u(x,t))_x = 0, \quad x \in R, \quad t > 0, \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in R. \tag{2}$$

The previous equation arises in modeling a diverse set of physical phenomena, ranging from shock waves to three-phase flows in porous media. Shock waves are modeled by nonlinear hyperbolic partial differential equations [8-10] and occur in explosions, traffic flow, glacier waves, airplanes breaking the sound barrier, and so on. To solve Eq. (1), one can use the method of characteristic curves. The characteristic system associated with the this equation is, [24],

$$\frac{dx}{f'(u)} = dt = \frac{du}{0}, \tag{3}$$

which is associated with the defining ordinary differential equation

$$\frac{dx}{dt} = f'(u). \tag{4}$$

The characteristic Eq.(4) is nonlinear in the unknown function $u(x,t)$ itself ; and each solution $u(x,t)$ of (1) will give a different set of characteristics. Upon integrating the characteristic system (3), one obtains

$$u_1(x,t) = u(x,t), \quad u_2(x,t) = x - f'(u(x,t)), \tag{5}$$

which give the following general solution

$$u(x,t) = x - F\left(x - f'(u)\, t\right). \tag{6}$$

By using the initial conditions at $t = 0$, one obtains

$$u(x,0) = u_0(x) = F(x). \tag{7}$$

Therefore for sufficiently small t, the general solution is given by

$$u(x,t) = u_0\left(x - f'(u)\, t\right), \tag{8}$$

and this solution is valid as long as the condition

$$1 + \frac{u_0\left(x - f'(u)\, t\right)}{dx}\; f''(u)\, t > 0 \tag{9}$$

is satisfied. Moreover, if the left-hand side of (7) approaches zero, the solution curve undergoes a discontinuity and a shock or shocks are developed.

The basic difficulty associated with this approach is that the solution is given implicitly by Eq. (6), which invokes a need for other reliable techniques to solve the shock wave Eq. (1). In fact Allan and Al-Khaled [4] have devised an analytical approach to solution of this equation by Adomian's decomposition method. Inspired and motivated by the ongoing research in this area, we shall apply the homotopy analysis method to obtain an approximation to the solution of this equation. The reached numerical results explicitly reveal a reliability of the proposed algorithm.

2. The Homotopy Analysis Method

In applying the HAM [15-19, 21-23] to solve the shock wave initial value problem, IVP, (1-2), we consider the differential equation

$$N[u(x,t)] = 0, \tag{10}$$

in which N is a nonlinear operator to construct the following zeroth-order deformation :

$$(1-q)\, L(U(x,t;q) - u_0(x,t)) = q\, \hbar\, H(x,t)\, N(U(x,t;q)), \tag{11}$$

with $q \in [0,1]$ as an embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function of the independent variables $x,\ t$ and q.
Obviously, when $q = 0$ and $q = 1$, the following notation

$$U(x,t;0) = u_0(x,t), \quad U(x,t;1) = u(x,t). \tag{12}$$

will respectively hold.
Next we expand $U(x,t;q)$ in Taylor series, with respect to the q parameter, as

$$U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{13}$$

where

$$u_m = \frac{1}{m!} \frac{\partial^m U(x,t;q)}{\partial^m q}\Bigg|_{q=0} . \tag{14}$$

Assuming that the auxiliary linear operator, the initial guess, $u_0(x,t)$, the auxiliary parameter $\hbar$ and the auxiliary function $H(x,t)$ are selected so as to make the series (13) convergent at $q = 1$, then due to (12) we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{15}$$

Let us define then the vector

$$\vec{u}_n(x,t) = (u_0(x,t), u_1(x,t), \ldots, u_n(x,t)), \tag{16}$$

and

$$\chi_m = \begin{cases} 0 & m \leqslant 1, \\ 1 & m > 1, \end{cases} \tag{17}$$

to differentiate (11) m times with respect to the embedding parameter q. Finally set $q = 0$ and divide by m! to obtain the so-called mth-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}), \tag{18}$$

in which

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N(U(x,t;q))}{\partial^{m-1} q} \right|_{q=0}. \tag{19}$$

For practical computational applications we may approximate the HAM solution (15) by the truncated series:

$$\phi_m(x,t) = \sum_{k=0}^{m-1} u_k(x,t). \tag{20}$$

3. The Shock Wave Equation

In this section we study the analytical solution of the shock wave equation for most gases, as given by [3, 11, 14],

$$u_t(x,t) + \left(\frac{1}{c_0} - \frac{\gamma+1}{2} \frac{u}{c_0^2} \right) u_x = 0, \quad (x,t) \in R \times [0,T], \tag{21}$$

where c_0, γ are constants and γ is the specific heat. In the case under current consideration for shock waves in air, we need to take $c_0 = 2$ and $\gamma = \frac{3}{2}$. Eq.(21) becomes correspondingly

$$u_t(x,t) + \left(\frac{1}{2} - \frac{5}{16} u \right) u_x = 0, \quad (x,t) \in R \times [0,T], \tag{22}$$

and is subject to the initial condition

$$u(x,0) = e^{-x^2/2}. \tag{23}$$

If $c_0 >> \frac{1}{2}(\gamma+1)$ it is shown in [6] that an exact series solution to the previous IVP exists the form

$$u(x,t) = \sum_{n=0}^{\infty} \frac{(-tB)^n}{(n+1)!} (n+1)^{n/2} H_n\left(\sqrt{n+1}\right) e^{-(x-t/2)^2(n+1)/2}, \tag{24}$$

where $B = (\gamma+1)/2c_0^2$ and $H_n(.)$ is the Hermit polynomial of order n. This solution happens to be the same as

$$u(x,t) = e^{-(x-t/2)^2/2}\Big[1-\tfrac{5}{16}t(x-t/2)e^{-(x-t/2)^2/2} + \tfrac{25\,t^2}{512}[3((x-t/2)^2-1)]e^{-(x-t/2)^2/2} + \ldots\Big]. \tag{25}$$

Moreover, according to the existence condition, given by Eq. (9), the solution of Eq. (25) exists if

$$1 > \frac{5t}{8}\Big[x - \Big(\frac{1}{2} - \frac{5}{16}u\Big)t\Big]e^{-(x-(1/2-5/16u)t)^2/2}. \tag{26}$$

Simultaneously according to (11), the zero-order deformation will be

$$(1-q)L(U(x,t;q) - u_0(x,t)) = q\,\hbar\,H(x,t)\Big(U_t + \Big(\frac{1}{2} - \frac{5}{16}U\Big)U_x\Big). \tag{27}$$

Hence, starting with an initial approximation $u_0(x,t) = e^{-x^2/2}$ and choosing the auxiliary linear operator

$$L(U(x,t;q)) = \frac{\partial U(x,t;q)}{\partial t},$$

with the property

$$L(C) = 0,$$

where C is an integral constant, and choosing the auxiliary function to satisfy

$$H(x,t) = 1,$$

we end up with the mth-order deformation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)R_m(\vec{u}_{m-1}),$$

in which

$$R_m(\vec{u}_{m-1}) = \frac{\partial(u_{m-1})}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(u_{m-1}) - \frac{5}{16}\left(\sum_{i=0}^{m-1} u_i\frac{\partial}{\partial x}(u_{m-1-i})\right). \tag{28}$$

The solution of the mth-order deformation equations (28) for $m \geqslant 1$ becomes

$$u_m(x,t) = \chi_m\,u_{m-1}(x,t) + \hbar\,L^{-1}[R_m(\vec{u}_{m-1})]. \tag{29}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = e^{-x^2/2},$$

$$u_1(x,t) = \hbar tx\Big[\frac{5}{16}e^{-x^2} - \frac{1}{2}e^{-x^2/2}\Big],$$

$$u_2(x,t) = \hbar tx\Big[\tfrac{5}{16}e^{-x^2} - \tfrac{1}{2}e^{-x^2/2}\Big] + \hbar^2 tx\Big[\tfrac{5}{16}e^{-x^2} - \tfrac{1}{2}e^{-x^2/2}\Big] +$$
$$\frac{\hbar^2 t^2}{512}e^{-x^2}\Big[-25e^{-x^2} + 40e^{-x^2/2} - 64x + 40xe^{-x^2/2} + 75x^2e^{-x^2} - 80x^2e^{-x^2/2}\Big], \qquad (30)$$

....

and so on. Obviously then the HAM series solution (for $\hbar = -1$) is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots\ldots$$

$$= e^{-x^2/2} - tx\Big[\tfrac{5}{16}e^{-x^2} - \tfrac{1}{2}e^{-x^2/2}\Big] +$$
$$\frac{t^2}{512}e^{-x^2}\Big[-25e^{-x^2} + 40e^{-x^2/2} - 64x + 40xe^{-x^2/2} + 75x^2e^{-x^2} - 80x^2e^{-x^2/2}\Big] + \ldots\ldots \qquad (31)$$

Theorem 3.1. *Let $u_m(x)$ satisfy Eq. (29) under the definition (28). If the series $\sum_{m=0}^{\infty} u_m(x)$ converges, it must be an exact solution to the IVP (22)- (23).*

Proof. Covergence of this series allows for $s = \sum_{m=0}^{\infty} u_m(x)$, and satisfaction of

$$\lim_{m\to\infty} u_m(x) = 0$$

is necessary for the existence of s. Make use then of (18) to write

$$\sum_{m=1}^{\infty} \hbar\, H(x,t)\, R_m(\vec{u}_{m-1}) = \lim_{n\to\infty} \sum_{m=1}^{n} L[u_m(x,t) - \chi_m u_{m-1}(x,t)]$$

$$= L\left\{\lim_{n\to\infty} \sum_{m=1}^{n}[u_m(x,t) - \chi_m u_{m-1}(x,t)]\right\} = L\left\{\lim_{n\to\infty} \sum_{m=1}^{n} u_n(x,t)\right\} = 0.$$

Since $\hbar \neq 0$ and $H(x) \neq 0$, the last relation becomes

$$\sum_{m=1}^{\infty} R_m(\vec{u}_{m-1}) = 0.$$

Finally, Substitution of (28) in the above expression yields

$$\sum_{m=1}^{\infty} R_m\big(\vec{u}_{m-1}\big) = \frac{\partial(u_{m-1})}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(u_{m-1}) - \frac{5}{16}\left(\sum_{i=0}^{m-1} u_i \frac{\partial}{\partial x}(u_{m-1-i})\right) = 0.$$

This ends the proof. ■

4. An HPM Solution

To solve the IVP (22-23) by the homotopy perturbation method (HPM), we construct the following homotopy:

$$\left(\frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t}\right)=p\left(\frac{5}{16}u\frac{\partial u}{\partial x}-\frac{1}{2}\frac{\partial u}{\partial x}-\frac{\partial u_0}{\partial t}\right), \tag{32}$$

Assume the solution of Eq.(32) in the form:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots.. \tag{33}$$

Substituting (33) into Eq.(32) and collecting terms of the same power of p gives

$$p^0:\quad \frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0,$$

$$p^1:\quad \frac{\partial u_1}{\partial t}=\frac{5}{16}u_0\frac{\partial u_0}{\partial x}-\frac{1}{2}\frac{\partial u_0}{\partial x}-\frac{\partial u_0}{\partial t},$$

$$p^2:\quad \frac{\partial u_2}{\partial t}=\frac{5}{16}u_0\frac{\partial u_1}{\partial x}+\frac{5}{16}u_1\frac{\partial u_0}{\partial x}-\frac{1}{2}\frac{\partial u_1}{\partial x},$$

$$p^3:\quad \frac{\partial u_3}{\partial t}=\frac{5}{16}u_0\frac{\partial u_2}{\partial x}+\frac{5}{16}u_1\frac{\partial u_1}{\partial x}+\frac{5}{16}u_2\frac{\partial u_0}{\partial x}-\frac{1}{2}\frac{\partial u_2}{\partial x},$$

$\ldots. : \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The given initial values admit the use of

$$u_0(x)=e^{-x^2/2},$$

and lead to the solution components that follow.

$$u_0(x,t)=e^{-x^2/2},$$

$$u_1(x,t)=tx\left[-\frac{5}{16}e^{-x^2}+\frac{1}{2}e^{-x^2/2}\right],$$

$$u_2(x,t)=\frac{t^2}{512}e^{-x^2}\left[-25\,e^{-x^2}+40\,e^{-x^2/2}-64\,x+40\,xe^{-x^2/2}+75\,x^2e^{-x^2}-80\,x^2e^{-x^2/2}\right],$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and so on. Hence, the HPM series solution is

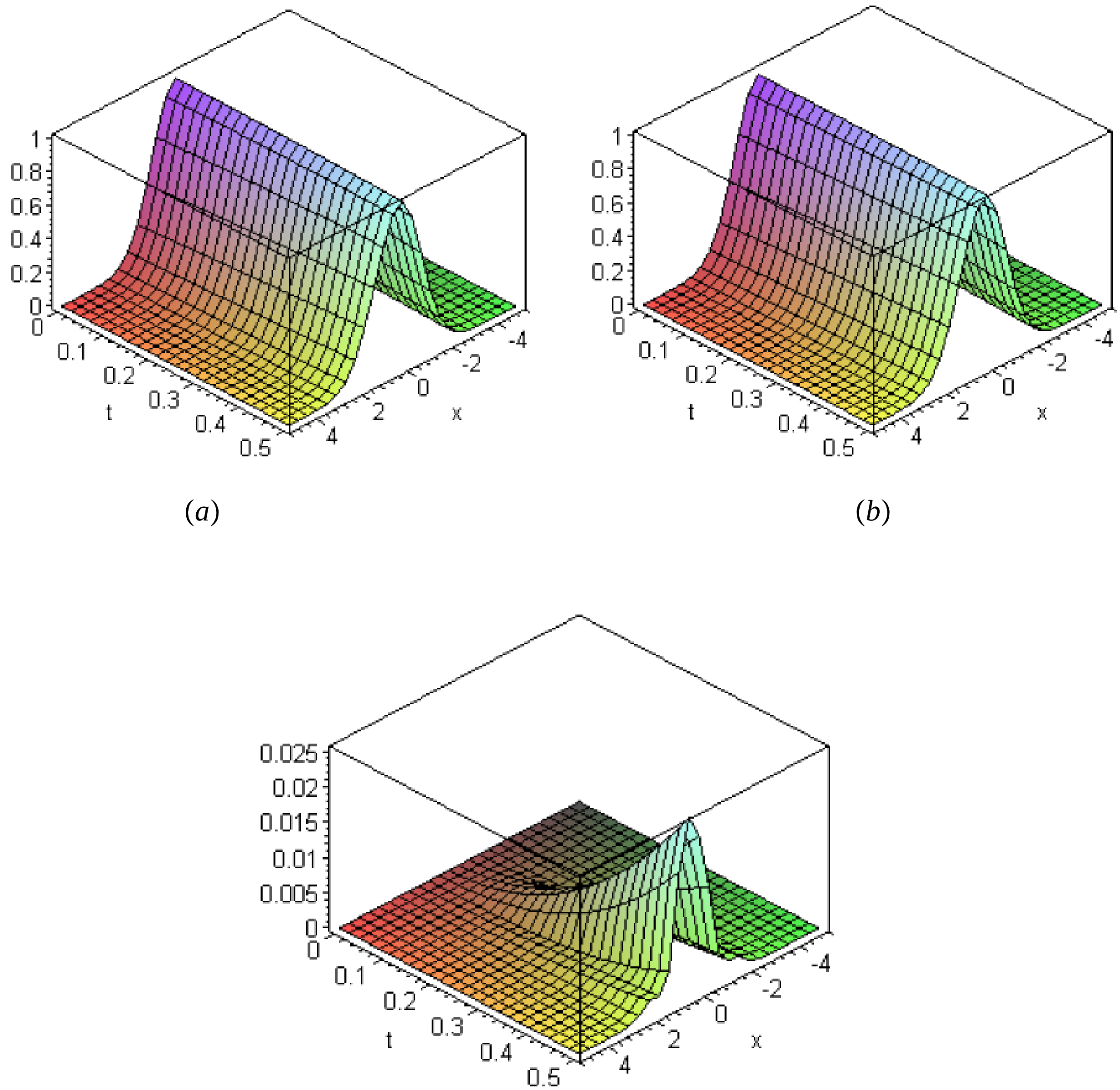
$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\ldots\ldots.$$

$$\begin{aligned}&=e^{-x^2/2}-tx\left[\tfrac{5}{16}e^{-x^2}-\tfrac{1}{2}e^{-x^2/2}\right]+\\ &\frac{t^2}{512}e^{-x^2}\left[-25\,e^{-x^2}+40\,e^{-x^2/2}-64\,x+40\,xe^{-x^2/2}+75\,x^2e^{-x^2}-80\,x^2e^{-x^2/2}\right]+\ldots.,\end{aligned} \tag{34}$$

which coincides with the (31) HAM solution .

5. Results and Discussion

Based on the HAM, we have constructed the solution of Eq. (20), using the recurrence relation given by Eq. (17). To demonstrate the accuracy of the method in solving the IVP (22)-(23), we have calculated the first few terms (n= 6) of the solution $u(x,t)$ using Eq. (25) derived in [6] and compared the results with the solution based on Eq. (17).



(a) (b)

(c)

Fig. 1. Surface plot of $u(x,t)$ for Eq.(22-23) : $-5 \leqslant x \leqslant 5, \quad \textit{and} \quad 0 \leqslant t \leqslant 0.5$
exact solution, (b) sixth order homotopy analysis approximation, (c) absolute error

Fig. 1(c) represents the error which is defined at any point as $|u(x,t) - u_{appr}(x,t)|$ where $u(x,t)$ is given by Eq. (25). It indicates that the results of the reported method are getting very close to the exact solution even when a small number of terms are used. The error can be reduced by adding more terms to the homotopy analysis series of Eq. (15). Figs. 1(a)-2(a) and 1(b)-2(b) represent respectively the surface plot of the wave velocity $u(x,t)$and $u_{appr}(x,t)$ for $-5 \leqslant x \leqslant 5$, and for $0 \leqslant t \leqslant 0.5$ together with the contour plot of that surface. They clearly show that the shock wave is traveling smoothly in this range of t.

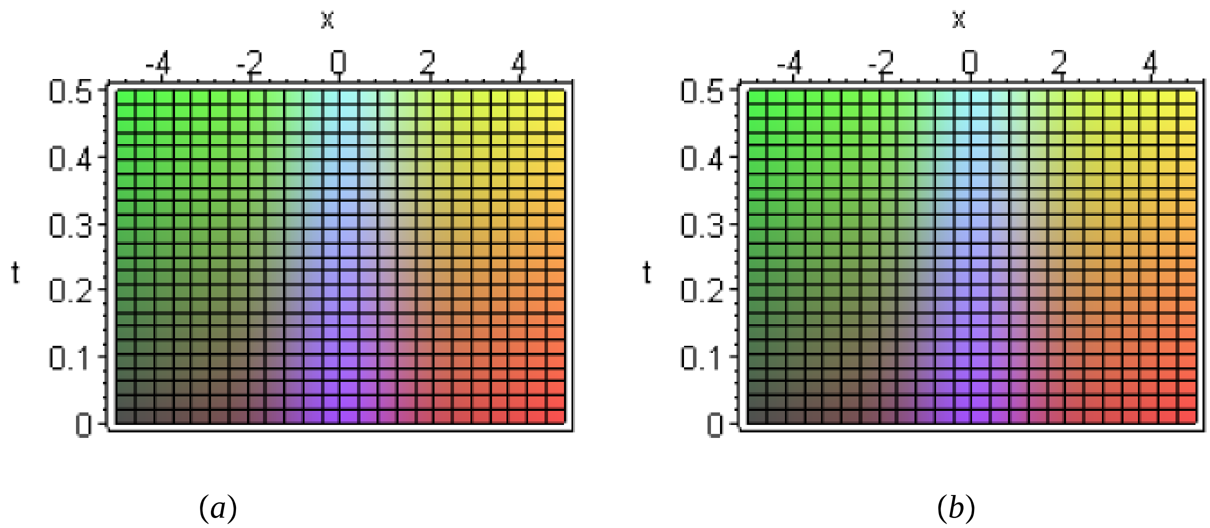


Fig. 2. Contour plot of $u(x,t)$ for Eq.(22-23) : $-5 \leqslant x \leqslant 5$, *and* $0 \leqslant t \leqslant 0.5$
exact solution, (b) sixth order homotopy analysis approximation

From these results we may conclude that the semi-analytical homotopy analysis method for the KdV equation gives remarkable accuracy in comparison with the analytical solution (25).

6. Conclusion

In this work, our main concern has been to study the dynamics of a shock wave development in air. An approximation to the analytical solution for the pertaining nonlinear IVP was obtained in the range $[0,\infty)$ of t by applying the HAM and symbolic calculations. A comparison of the numerical results presented in this article with the results of [6] suggests that the HAM is accurate, reliable and user friendly.

Acknowledgments.
The authors would like to thank one of the referees for a number of very helpful comments.

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Article history : Submitted December, 25, 2009; Accepted May, 29, 2010.