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Quasi-exact Numerical Evaluation of Synthetic Collateralized Debt Obligations Prices

W. HÜRLIMANN

FRSGlobal Switzerland, Seefeldstrasse 69, CH-8008 Zurich, Switzerland
E-mail: werner.huerlimann@frsglobal.com

Abstract. *The most popular approach to synthetic collateralized debt obligations (CDO) pricing uses factor models in the conditional independence framework, which were first introduced by Vasicek to estimate the loan loss distribution of a pool of loans. Efficient methods for evaluating the loss distributions of synthetic CDO's are important for both pricing and risk management purposes. In the framework of the one-factor Gaussian copula model, we propose an approximate but quasi-exact numerical recursive evaluation using pseudo compound Poisson distributions. For the sake of illustration and comparison we have computed a number of more or less complex cases, whose approximations turn out to be highly accurate in all considered examples.*

Key words : Synthetic CDO's, One-factor Vasicek Model, Pseudo Compound Poisson Distribution, Recursive Algorithm.

AMS Subject Classifications : 60E10, 62E17, 62P05, 65C20, 65Q99

1. Introduction

Collateralized debt obligations (CDO) are among those structured financial products, which had an important impact during the sub-prime mortgage crisis. The Wikipedia entry http://en.wikipedia.org/wiki/Subprime_mortgage_crisis claims that Merrill Lynch's large losses in 2008 were attributed in part to the drop in value of its un-hedged portfolio of CDO's after AIG ceased offering credit default swaps (CDS) on Merrill's CDO's. Knowledge of the risk characteristics of synthetic CDO's is important for understanding the nature and magnitude of credit risk transfer. In particular, efficient methods for evaluating the loss distributions of synthetic CDO's are important for both pricing and risk management purposes. Recall some known methods, which can be divided into several groups as follows:

Analytical and semi-analytical methods. Through simplification of the pricing models, analytical or at least semi-analytical pricing expressions can be obtained. Factor models, such as the reduced-form model proposed by [30] and the structural model proposed by [38], [39], [40] (see also [32], [5], Section 2.5.1, [15]) are widely used in practice to obtain analytic or semi-analytic formulas to price synthetic CDO's efficiently. For a comparative analysis of different factor models, we refer to [6]. Further interesting analytical models in this area include [28] and [34], which use normal inverse Gaussian distributions, and [4], which extends Vasicek's asymptotic model to general non-normal systematic risk factors.

Monte Carlo method. From a computational point of view, Monte Carlo simulation is the last resort because of its inefficiency, despite its flexibility, and is not discussed further.

Exact evaluation methods. The available numerical methods assume that the loss-given-defaults of all obligors are integer multiples of a properly chosen monetary unit (common lattice assumption). Exact methods have been given by [2], [30] and [20]. A discussion of these methods, a multi-state extension, as well as a stable and efficient reformulation of the Hull and White method are found in [26].

Quasi-exact evaluation method. Approximate numerical evaluation of the pool's loss distribution is possible. An example is the compound Poisson approximation by [7]. Following [26] improved and almost exact accuracy can be obtained using the so-called pseudo compound Poisson approximations by [22] in the form proposed in [17], [18]. The present mathematical specification, written in the spirit of [13], is devoted to the latter quasi-exact numerical method.

The exposé is organized as follows. Section 2 recalls the pricing model for synthetic CDO's. Section 3 presents the approximate and quasi-exact evaluation using pseudo compound Poisson distributions and Section 4 illustrates its use at some simple examples.

2. Pricing Model

2.1. Fair spread

A synthetic collateralized debt obligation, or synthetic CDO, is a transaction that transfers the credit risk on a reference portfolio of assets. The reference portfolio in a synthetic CDO is made up of credit default swaps or CDS's. A synthetic CDO is classified as a credit derivative. Much of the risk transfer that occurs in the credit derivatives market is in the form of synthetic CDO's. Understanding the risk characteristics of synthetic CDO's is important for understanding the nature and magnitude of credit risk transfer (see [14]).

Consider a synthetic CDO tranche of size S with an attachment point ℓ, a threshold that determines whether some of the pool losses shall be absorbed by this tranche. If the realized losses of the pool are less than ℓ, then the tranche will not suffer any loss, otherwise it will absorb losses up to its size S. The threshold $S+\ell$ is called the *detachment point* of the tranche. Assume there are *m names* in the pool. For name $k \in \{1,\dots,m\}$, its *notional value* and the *recovery rate* of the notional value of the reference asset are denoted by N_k and R_k, respectively. Then the *loss-given-default* or the recovery-adjusted notional value of name k, is $LGD_k = N_k \cdot (1-R_k)$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ be the set of *premium dates*, with T

denoting the *maturity date* of the CDO tranche. Assume that the interest rates are deterministic. Then the set of (risk-free) *discount factors* for the given payment dates, denoted by $D_1, D_2, \dots, D_n$, are deterministic. Let L_i^P be the pool's cumulative losses up to time $t_i, i \in \{1, \dots, n\}$. Then the losses absorbed by the specified tranche up to time t_i, denoted by L_i, is $L_i = min\{(L_i^P - \ell)_+, S\}$, where $x_+ = max\{x, 0\}$. The function $p(L_i^P; S, \ell) = min\{(L_i^P - \ell)_+, S\}$, is called the *payoff function* of the specified tranche. In actuarial science a similar payoff function is used to define the limited stop-loss reinsurance, where L_i^P represents the cumulative claims up to the i –th claim, with the difference that the number of claims n up to the maturity date T of the reinsurance contract is random.

Assume that the fair spread for the tranche is a constant s per annum. The two important quantities to be determined in synthetic CDO tranche valuation are the present value of the *default leg* (the expected losses of the tranche over the life of the contract), called *contingent*, and the present value of the *premium leg* (the expected premiums that the tranche investor will receive over the life of the contract), called *fee*. We use the definitions and relationships

default leg: $DL = \sum_{i=1}^n D_i(L_i - L_{i-1})$

premium leg: $PL = s \cdot \sum_{i=1}^n D_i \Delta_i (S - L_i),\ \Delta_i = t_i - t_{i-1}$

contingent: $PV(DL) = \sum_{i=1}^n D_i \mathfrak{E}[L_i - L_{i-1}],\ \mathfrak{E}[L_0] = 0$

fee: $PV(PL) = s \cdot \sum_{i=1}^n D_i \Delta_i (S - \mathfrak{E}[L_i])$

The *market-to-market* value of the tranche to the tranche investor today is equal to

$$MTM = fee - contingent$$

The *fair spread* solves the pricing equation $MTM = 0$, and is given by

$$s = \frac{\sum_{i=1}^n D_i \mathfrak{E}[L_i - L_{i-1}]}{\sum_{i=1}^n D_i \Delta_i (S - \mathfrak{E}[L_i])}\ ,\ \mathfrak{E}[L_0] = 0. \tag{1}$$

With (1) the valuation problem is reduced to the computation of the expected cumulative losses $\mathfrak{E}[L_i], i = 1, \dots, n$. In order to compute these expectations, one has to specify the default processes for each of the names and the correlation structure of the default events.

2.2. **One-factor model**

The most popular approach to synthetic CDO pricing uses factor models in the conditional independence framework. They were first introduced by [38] to estimate the loan loss distribution of a pool of loans. We will use a one-factor model.

Let T_k be the random default time of name $k \in \{1, \dots, m\}$ and assume that the *risk-neutral default probabilities* $q(k,i) = \mathfrak{P}(T_k < t_i), i = 1, \dots, n, k = 1, \dots, m$, are available as input. The latter quantities can be estimated from CDS single-name spreads (e.g. [12], [19], [3]). The dependence structure of the default times is determined by the *creditworthiness indices* Y_k through a one-factor copula and are defined by

$$Y_k = \sqrt{\rho_k} X + \sqrt{1 - \rho_k} Z_k\ , \tag{2}$$

with

X : systematic risk factor

Z_k : idiosyncratic factors

$\rho_k^2 \in (0,1)$: correlation factors

One assumes that the Z_k's are mutually independent and also independent of X. The risk-neutral default probabilities and the creditworthiness indices are related by the copula model

$$q(k,i) = \mathfrak{P}(Y_k < H_k(t_i)), i = 1,\dots,n, k = 1,\dots,m, \tag{3}$$

where $H_k(t_i)$ is the *default threshold* of the k –th name at time t_i. The copula model was first introduced by [32] and then used in portfolio credit risk analyses, including synthetic CDO valuation, by [16], [20], [7], [30] and [36] among others.

One notes that the correlations of the default events are captured by the systematic risk factor X and conditional on a given value x of X, all default events are independent. If one assumes furthermore that X and Z_k follow standard normal distributions, then one obtains the so-called *one-factor Gaussian copula model*. In this standard model one has the relationships

(i) $H_k(t_i) = \Phi^{-1}(q(k,i))$

(ii) $Cov[Y_k, Y_j] = \sqrt{\rho_k \rho_j}$

(iii) $q(k, i \mid x) = \mathfrak{P}(Y_k < H_k(t_i) \mid X = x) = \Phi\left(\frac{\Phi^{-1}(q(k,i)) - \sqrt{\rho_k} X}{\sqrt{1-\rho_k}} \right)$

where $\Phi(z)$ is the standard normal distribution, and (iii) represents conditional risk-neutral default probabilities.

Remarks 2.1. The one-factor Gaussian copula model can be extended in various ways:

- If X is a random vector, one obtains a multi-factor copula model
- If X and Z_k follow Student–$t$distributions with different degrees of freedom, one obtains the double–t copula model in [20]
- If X and Z_k follow normal inverse Gaussian distributions one obtains models of the type considered in [28] and [34]
- Further generalizations are found in [6], [4] and [1]

In the above conditional independence framework, the expected cumulative tranche losses $\mathfrak{E}[L_i], i = 1,\dots,n$ can be computed as

$$\mathfrak{E}[L_i] = \int_{-\infty}^{\infty} \mathfrak{E}[L_i \mid X = x]\, d\Phi(x)\ , \tag{4}$$

where $\mathfrak{E}[L_i \mid X = x] = \mathfrak{E}[min\{(L_i^P - \ell)_+, S\} \mid X = x]$ is the expectation of the tranche loss L_i conditional on $X = x$. Clearly one has

$$L_i^P = \sum_{k=1}^{m} LGD_k \bullet I\{Y_k < \Phi^{-1}(q(k,i))\}, \tag{5}$$

where the random indicators $I\{Y_k < \Phi^{-1}(q(k,i))\}$ are mutually independent conditional on X. With (5) the valuation problem is further reduced to the computation of the conditional expected cumulative losses $\mathfrak{E}[L_i \mid X = x]$, $i = 1,\dots,n$. A quasi-exact recursive algorithm for this is developed in the next Section.

Remark 2.2. An alternative way to evaluate $\mathfrak{E}[L_i \mid X = x]$ consists in approximating the CDO tranche payoff function $p(L_i^P; S, \ell) = min\{(L_i^P - \ell)_+, S\}$ by a sum of exponentials over the interval $[0,\infty)$ as proposed in [24], [25].

3. Recursive Evaluation Via Pseudo Compound Poisson Distributions

For convenience the systematic risk factor is fixed at some value $X = x$. Random sums of the type (5) with mutually independent terms are well-known in actuarial science under the heading of "individual model of risk theory". Methods to evaluate its distribution function have been designed by many authors including [29], [17], [8], [9], [10], [21], [22], [23], [37]. The main basic idea consists to consider approximations to the characteristic function of (5) and develop recursive algorithms for the evaluation of the corresponding distribution functions. By adequate choice of the approximation, the evaluation can be made as accurate as desired.

Conditional on $X = x$ the characteristic function of (2.5) is given by

$$\phi(t) = \prod_{k=1}^{m} \phi_k(t), \ \ \phi_k(t) = \exp\{ln[1 + c_k \cdot (e^{it \cdot LGD_k} - 1)]\}, \tag{6}$$

where for simplicity of notation the shortcut $c_k = q(k, i \mid x)$ is used. References [17] and [18], Chapter 4, define the J–th order approximation of (6) for small c_k by truncating the logarithmic expansion $ln(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j$ at the J–th term to get the expression

$$\phi^{(J)}(t) = \exp\left\{\sum_{k=1}^{m} \sum_{j=1}^{J} \frac{(-1)^{j+1}}{j} [c_k \cdot (e^{it \cdot LGD_k} - 1)]^j\right\}, \ J = 1, 2, 3, \ldots.. \tag{7}$$

For $J = 1$ (7) can be rewritten as

$$\phi^{(1)}(t) = \exp\left\{\lambda_1 \, (\psi_1(t) - 1)\right\}, \ \lambda_1 = \sum_{k=1}^{m} c_k \, , \ \psi_1(t) = \frac{1}{\lambda_1} \sum_{k=1}^{m} c_k \, e^{it \cdot LGD_k} \, , \tag{8}$$

which is the characteristic function of a *compound Poisson* distributed random variable with Poisson parameter λ_1 and probability function

$$h_1(y) = \frac{1}{\lambda_1} \cdot \sum_{LGD_k = y} c_k \, , \ y = 1, 2, 3, \ldots.. \tag{9}$$

Similarly, for all $J \geq 2$, (7) can also be rewritten in the form $\phi^{(J)}(t) = \exp\left\{\lambda_J \, (\psi_J(t) - 1)\right\}$, which corresponds in the terminology of [22] to the characteristic function of a *pseudo compound Poisson* distributed random variable with Poisson parameter λ_J and pseudo probability function $h_J(y)$. The approximations of smaller order $J = 2, 3, 4$ are ([18], p.79-80):

<u>$J = 2$</u>

$$\lambda_2 = \sum_{k=1}^{m} c_k \cdot (1 + \tfrac{1}{2} c_k)$$

$$h_2(y) = \tfrac{1}{\lambda_2} \cdot \left[\sum_{LGD_k = y} c_k \cdot (1 + c_k) - \tfrac{1}{2} \cdot \sum_{2 \cdot LGD_k = y} c_k^2\right], \ y = 1, 2, \ldots$$

<u>$J = 3$</u>

$$\lambda_3 = \sum_{k=1}^{m} c_k \cdot (1 + \tfrac{1}{2} c_k + \tfrac{1}{3} c_k^2)$$

$$h_3(y) = \tfrac{1}{\lambda_3} \cdot \left[\sum_{LGD_k = y} c_k \cdot (1 + c_k + c_k^2) - \sum_{2 \cdot LGD_k = y} c_k^2 \cdot (\tfrac{1}{2} + c_k) + \tfrac{1}{3} \cdot \sum_{3 \cdot LGD_k = y} c_k^3\right],$$

$y = 1, 2, \ldots$

<u>$J = 4$</u>

$\lambda_4 = \sum_{k=1}^{m} c_k \cdot (1 + \frac{1}{2} c_k + \frac{1}{3} c_k^2 + \frac{1}{4} c_k^3)$

$h_4(y) = \frac{1}{\lambda_4} \cdot \Big[\sum_{LGD_k = y} c_k \cdot (1 + c_k + c_k^2 + c_k^3) - \sum_{2 \cdot LGD_k = y} c_k^2 \cdot (\frac{1}{2} + c_k + \frac{3}{2} c_k^2) + \frac{1}{3} \cdot \sum_{3 \cdot LGD_k = y} c_k^3 \cdot (\frac{1}{3} + c_k) - \frac{1}{4} \cdot \sum_{4 \cdot LGD_k = y} c_k^4 \Big], \; y = 1, 2, \ldots$

At this stage some mathematical comments are in order. The functions $h_J(y)$ do not define true probability measures but only signed measures. The conditions under which a pseudo compound Poisson distribution with Poisson parameter λ and pseudo probability function $h(y)$, $y = 1, 2, \ldots$ defines a true probability distribution have been identified in [31]. According to [33], p.252, and [27], p.356, this is the case provided a negative value $h(y) < 0$ is preceded by a positive value and followed by at least two positive values. This criterion is not always fulfilled in Example 4.4. It is fulfilled for $J = 1, 3$ but not for $J = 2, 4$. However, the latter anomaly does not disturb the obtained results. Another remarkable property of Hipp's pseudo compound Poisson approximation has been derived in [11]. The distribution function corresponding to the J−th order approximation of (6) has the same first J moments as the original distribution corresponding to (6). In particular, the 4 −th order approximation fits the mean, variance, skewness and kurtosis of the original distribution. More importantly, the probability function $f(z), z = 0, 1, 2, \ldots$ of a pseudo compound Poisson distribution with Poisson parameter λ and pseudo probability function $h(y), y = 1, 2, \ldots$ can be evaluated using the simple Panjer recursion formula (e.g. [22], formula (1.5)):

$$f(0) = e^{-\lambda}, \quad z \cdot f(z) = \lambda \cdot \sum_{y=1}^{z} y \cdot h(y) \cdot f(z - y), \; z = 1, 2, \ldots \tag{10}$$

As shown in [35], this recursive algorithm is numerically stable. Finally, increasing the order of approximation to infinity guarantees arbitrary accuracy and convergence to the probability distribution corresponding to (6). Let $F(z)$, $F^{(J)}(z)$ be the distribution of (6) and its J−th order approximation. According to [18], p.80, one has the error bound

$$|F(z) - F^{(J)}(z)| \le e^{\varepsilon} - 1, \; \varepsilon = \sum_{k=1}^{m} \varepsilon_k, \; \varepsilon_k = \frac{1}{J+1} \frac{(2c_k)^{J+1}}{1 - 2c_k}, \; c_k < \frac{1}{2}. \tag{11}$$

In particular, letting $J \to \infty$ this shows convergence of the chosen approximation method.

4. Numerical Examples

So far we have developed a convergent recursive algorithm for the evaluation of the probability function associated to the pool's cumulative loss (5) conditional on a given value of the systematic risk factor. For fixed $J \in \{1, 2, 3, 4\}$, denote by $f_i^{P,J}(z \mid x)$ the J−th order approximation of the conditional probability function $\mathfrak{P}(L_i^P = z \mid X = x)$ associated to (5), which has been calculated using the recursive algorithm (10) with Poisson parameter $\lambda = \lambda_J$ and pseudo probability function $h(y) = h_J(y)$ as specified in Section 3. To obtain the cumulative tranche losses (4) we first calculate the unconditional probability function of (5) via numerical integration as follows:

$$f_i^{P,J}(z) = \int_{-\infty}^{\infty} f_i^{P,J}(z \mid x) \, d\Phi(x) \approx \frac{\triangle}{N} \cdot \sum_{r=-N}^{N} f_i^{P,J}(z \mid \triangle \cdot \tfrac{r}{N}) \, \varphi(\triangle \cdot \tfrac{r}{N}), \tag{12}$$

where $\varphi(t) = \Phi'(t)$ is the standard normal probability density. In our numerical examples the choice $\Delta = 5, N = 500$, has been appropriate. Associated to (12) we compute the probability distribution function setting

$$f_i^{P,J}(z) = \sum_{y=0}^{z} f_i^{P,J}(y)\ ,\ z = 0,1,2,3,\ldots. \qquad (13)$$

and the stop-loss transform $SL_i^{P,J}(z) = \mathfrak{E}^J[(L_i^P - z)_+]$ via the recursion

$$SL_i^{P,J}(0) = \mathfrak{E}[L_i^P] = \sum_{k=1}^{m} q(k,i)\, LGD_k\ ,\ \ SL_i^{P,J}(z+1) = SL_i^{P,J}(z) - 1 + L_i^{P,J}(z). \qquad (14)$$

The J–th order approximation of the expected cumulative tranche losses (2.4) is then obtained by setting

$$\mathfrak{E}^J[L_i] = \mathfrak{E}^J[min\{(L_i^P - \ell)_+, S\}] \ = \ SL_i^{P,J}(\ell) - SL_i^{P,J}(\ell + S). \qquad (15)$$

By inserting the obtained values into (1) one gets J–th order approximations of the fair spreads of synthetic CDO's, which with increasing approximation order will convergence to the exact fair spread. For the sake of illustration and comparison we have computed a number of more or less complex cases.

Example 4.1. Completely homogeneous pool.

Suppose that there are $m = 100$ names in the pool, each with identical loss-given-default $LGD_k = N_k \cdot (1 - R_k) = 1$. Let $t_i = i, i = 1,\ldots,5$ be the premium dates, $T = 5$ the maturity date. Each name in the pool has risk-neutral default probabilities $q(k,i) = q(i) = 1 - e^{-0.01 \cdot i}$, $i = 1,\ldots,5$, and let $\rho_k = \rho = 30\%$ be the identical correlation factors of the one-factor Gaussian copula model. The discount factors are based on a risk-free flat interest rate of 5%. In this completely homogeneous situation the conditional probability function of (5) is exact binomially distributed such that

$$f_i^P(z \mid x) = \binom{m}{z} q^z(i \mid x)\, [1 - q(i \mid x)]^{m-z}\ ,\ z = 1,2,\ldots,m, \qquad (16)$$

with $q(i \mid x) = \Phi\left(\frac{\Phi^{-1}(q(i)) - \sqrt{\rho} \cdot X}{\sqrt{1-\rho}}\right)$. The approximation of order $J = 1$ is exact conditional Poisson distributed with parameter $\lambda_1 = m \cdot q(i \mid x)$. Moreover as $m \to \infty$ the large portfolio Vasicek limiting distribution holds such that

$$f_i^P(z) \to \Phi\left(\frac{\sqrt{1-\rho}\ \cdot \Phi^{-1}(\frac{z}{m})\ - \Phi^{-1}(q(i))}{\sqrt{\rho}}\right),\ z = 1,2,\ldots,m. \qquad (17)$$

The Table1 below summarizes the results of par spread calculation for three CDO tranches, an equity tranche between 0 and 3 defaults, a mezzanine tranche between 3 and 10 defaults and a senior tranche between 10 and (maximally) 100 defaults. A comparison of the results shows that the exact results up to the third decimal place are already attained for the pseudo compound Poisson approximation of order $J = 2$. The Poisson approximation of order $J = 1$ underestimates the spreads of the lower tranches while the Vasicek approximation is definitely not appropriate in this situation (overestimation of the equity and mezzanine tranches and underestimation of the senior tranche).

Table 1. Par spreads for the completely homogeneous pool.

	Par spread for different distributions					
CDO tranches	$J=1$	$J=2$	$J=3$	$J=4$	Exact	Vasicek
equity	21.794%	21.875%	21.876%	21.876%	21.876%	26.095%
mezzanine	6.004%	6.024%	6.024%	6.024%	6.024%	6.488%
senior	0.271%	0.269%	0.269%	0.269%	0.269%	0.201%

Example 4.2. Sub-pools with varying correlation factors and risk-neutral default probabilities.

Suppose that there are 5 sub-pools with 20 names in each sub-pool, each with identical loss-given-default $LGD_k = N_k \cdot (1 - R_k) = 1$. $t_i = i, i = 1, \ldots, 5$ be the premium dates, $T = 5$ the maturity date. Each name in the sub-pool $k \in \{1, \ldots, 5\}$ has risk-neutral default probabilities $q(k,i) = e^{-(0.005+0.005 \cdot k) \cdot i}$, $i = 1, \ldots, 5$, and correlation factors $\rho_k = 0.25 + 0.05 \cdot k$. There is a risk-free flat interest rate of 5%. In contrast to Example 1, the attachment and detachment of the CDO tranches are expressed in units of loss amounts. We consider three CDO tranches, an equity tranche between 0 and 10 loss units, a mezzanine tranche between 10 and 25 loss units, and a senior tranche between 25 and 100 loss units. Table 2 shows that the spreads of the pseudo compound Poisson approximation of order $J = 3$ are exact within three decimal places while the approximations of order $J = 2$ differ only slightly.

Table 2. Par spreads for the partially inhomogeneous pool of Example 4.2.

	Par spread for different distributions			
CDO tranches	$J=1$	$J=2$	$J=3$	$J=4$
equity	15.524%	15.585%	15.586%	15.586%
mezzanine	4.184%	4.207%	4.211%	4.211%
senior	0.408%	0.400%	0.399%	0.399%

Example 4.3. Sub-pools withub-pools with varying loss-given-defaults.

Suppose that there are 5 sub-pools with 20 names in each sub-pool, each name with loss-given-default $LGD_k = N_k \cdot (1 - R_k) = 1$. Let $t_i = i, i = 1, \ldots, 5$ be the premium dates, $T = 5$ the maturity date. Names in the sub-pools have identical risk-neutral default probabilities $q(k,i) = q(i) = 1 - e^{-0.01 \cdot i}$, $i = 1, \ldots, 5$, and correlation factors $\rho_k = \rho = 30\%$. There is a risk-free flat interest rate of 5%. As in Example 4.2 there are three CDO tranches, an equity tranche between 0 and 10 loss units, a mezzanine tranche between 10 and 25 loss units, and a senior tranche between 25 and 100 loss units. Table 3 shows that the spreads of the pseudo compound Poisson approximation of order $J = 3$ are exact up to three decimal places while the approximations of order $J = 2$ differ only slightly.

Table 3. Par spreads for the partially inhomogeneous pool of Example 4.3.

	Par spread for different distributions			
CDO tranches	$J = 1$	$J = 2$	$J = 3$	$J = 4$
equity	19.880%	19.964%	19.965%	19.965%
mezzanine	6.616%	6.645%	6.645%	6.645%
senior	1.174%	1.183%	1.187%	1.188%

Example 4.4. Inhomogeneous pool.

Let us combine the features of Example 4.2 and 4.3. Suppose that there are 5 sub-pools with 20 names in each sub-pool, each name with loss-given-default $LGD_k = N_k \cdot (1 - R_k) = k$, $k = 1,2,\ldots,5$. Let $t_i = i, i = 1,\ldots,5$ be the premium dates, $T = 5$ the maturity date. Each name in the sub-pool $k \in \{1,\ldots,5\}$ has risk-neutral default probabilities $q(k,i) = 1 - e^{-(0.005+0.005 \cdot k) \cdot i}$, $i = 1,\ldots,5$, and correlation factors $\rho_k = 0.25 + 0.05 \cdot k$. There is a risk-free flat interest rate of 5%. As in the Examples 4.2 and 4.3 there are three CDO tranches, an equity tranche between 0 and 10 loss units, a mezzanine tranche between 10 and 25 loss units, and a senior tranche between 25 and 100 loss units. Table 4 shows that the spreads of the pseudo compound Poisson approximation of order $J = 3$ are exact up to two decimal places while the approximations of order $J = 2$ differ only slightly from the exact values.

Table 4. Par spreads for the partially inhomogeneous pool of Example 4.4.

	Par spread for different distributions			
CDO tranches	$J = 1$	$J = 2$	$J = 3$	$J = 4$
equity	25.954%	26.087%	26.091%	26.091%
mezzanine	11.002%	11.078%	11.080%	11.080%
senior	3.002%	3.060%	3.076%	3.082%

The analyzed numerical examples allow for the following conclusions. The approximations of order J=3,4 yield quasi-exact spreads for CDO tranches. The approximation of order J=2 yields almost accurate spreads, which can be used in practical applications. The spreads from the compound Poisson approximation J=1 differ already too much to be reliable in general.

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