Journal of Numerical Mathematics and Stochastics, **4** (1) : 13-25, 2012 © ***JNM@S***
http://www.jnmas.org/jnmas4-2.pdf Euclidean Press, LLC
Online: ISSN 2151-2302

Solution of Fourth Order Boundary Value Problems by Numerical Algorithms Based on Nonpolynomial Quintic Splines

P. K. SRIVASTAVA[1], **M. KUMAR**[1], **and R. N. MOHAPATRA**[2]

[1]Department of Mathematics, M. Nehru National Institute of Technology, Allahabad-211004, India;
[2]Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA,
E-mail: panx.16j@gmail.com

Abstract. *A family of fourth and second-order accurate numerical schemes is presented for the solution of nonlinear fourth-order boundary-value problems (BVPs) with two-point boundary conditions. Non-polynomial quintic spline functions are applied to construct the numerical algorithms. This approach generalizes nonpolynomial spline algorithms and provides a solution at every point of the range of integration. Two numerical examples are given to illustrate the applicability and efficiency of the reported algorithms.*

Key words : Nonpolynomial Quintic Splines, Quintic Spline, Sixtic Splines, Nonlinear Two-Point Boundary Value Problems, Approximate Solutions.

AMS Subject Classifications : 41A15, 65L10

1. Introduction

Solution of fourth-order linear BVPs have been treated in a variety of ways including the use of nonpolynomial spline techniques. These success of these techniques motivate our interest in this work in nonlinear fourth-order BVPs involving the differential equation (DE)

$$y^{(4)}(x) = f(x,y), \quad a < x < b; \quad a,b,x \in R \tag{1}$$

subject to functional and second-order derivative boundary conditions (BCs)

$$y(a) = A_0, \; y''(a) = B_0, \; y(b) = A_1, \; y''(b) = B_1 \tag{2}$$

where A_i, B_i $i = 0,1$ are real finite constants. It is moreover assumed that $f(x,y)$ is real and continuous on $[a,b]$ with $\partial f/\partial y < 0$. A detailed discussion of the existence and uniqueness of the real valued function y(x) which satisfies (1) and (2) may be found in [20, 22, 23].

The approximate solution to this type of problems has recently been addressed by a few other methods. Amongst these we may mention the method of M. A. Ramadan et al. [1] which employs a quintic nonpolynomial spline for the numerical solution of linear fourth-order DEs

$$y^{(4)}(x) + f(x)y = g(x), \qquad x \in [a,b], \tag{3}$$

associated with a bending beam, subject to the BCs

$$y(a) - A_1 = y(b) - B_1 = y^{(2)}(a) - A_2 = y^{(2)}(b) - B_2 = 0. \tag{4}$$

Siraj-ul-Islam et al. developed in [2] a similar technique for a DE associated with obstacle, unilateral and contact problems of the type

$$y^{(4)} = \begin{cases} f(x) & a \le x \le c, \\ g(x)y(x) + f(x) + r & c \le x \le d, \\ f(x) & d \le x \le b, \end{cases} \tag{5}$$

subject to the boundary and the continuity conditions:

$$\begin{cases} y(a) = y(b) = A_1, \ y''(a) = y''(b) = A_2, \\ y(c) = y(d) = B_1, \quad y''(c) = y''(d) = B_2, \end{cases} \tag{6}$$

where f and g are continuous functuions on $[a,b]$ and $[c,d]$ respectively.
The same problem (5) has also been dealt with by S. S. Siddiqi et al. [3] when subjected to the following two cases of boundary conditions:

Case I

$$\begin{cases} y(a) = y(b) = \alpha_0, \ y''(a) = y''(b) = \alpha_1, \\ y(c) = y(d) = \alpha_2, \quad y''(c) = y''(d) = \alpha_3, \end{cases} \tag{7}$$

Case II

$$\begin{cases} y(a) = y(b) = \alpha_0, \ y'(a) = y'(b) = \alpha_4, \\ y(c) = y(d) = \alpha_2, \quad y'(c) = y'(d) = \alpha_5. \end{cases} \tag{8}$$

Parallel to these efforts, Siraj-ul-Islam et al. [4] and J. Rashidinia et al. [5] have extended the same technique to fourth-order BVPs associated with plate deflection theory, given by

$$[L + f(x)y(x)] = g(x), \quad L \equiv \frac{d^4}{dx^4}, \qquad a < x < b, \tag{9}$$

with BCs

$$y(a) = A_1, \quad y(b) = A_2, \qquad y'(a) = B_1, \qquad y'(b) = B_2. \tag{10}$$

The same problem (5) with BCs (8) was addressed in [6, 7] by E. A. Al-Said et al. On another note, R. A. Usmani developed in [8] finite difference methods for a continuous approximation to the solution of a two-point BVP involving a fourth order linear DE of the type (9) with the BCs

$$\begin{cases} y(a) - A_1 = y'(a) - A_2 = 0, \\ y^{(2)}(b) - B_1 = y^{(3)}(b) - B_2 = 0. \end{cases} \tag{11}$$

The singular version of (9), i.e.

$$y^{(4)} + p(x)y = q(x), \qquad -\infty < a < x < b < \infty, \tag{12}$$

subject to (10) has also been solved in [9] by the same R. A. Usmani. Moreover Loghmani and Alavizadeh [10] converted the problem in [9] to an optimal control problem to constructed an approximate solution as a combination of quartic B-splines. At the same time Rashidinia and Golbabaee [11] and Siddiqi and Akram [12] devised a difference scheme with quintic spline functions for this type of problem with the BCs (11). Also Van Daele et al. introduced in [13] a new second order method for solving the BVP (3) with the BCs involving first-order derivatives based on a nonpolynomial spline function.

Amongst further workers on this subject we may list R. A. Usmani [14] who developed and analyzed second-order and fourth-order convergent methods for the solution of a linear fourth-order two-point BVP (3) subject to (4) using a quartic polynomial spline function. Then E. H. Twizell [18], who used a two-grid fourth order method to get the solution of nonlinear BVP of the type (1) with BCs (2). This BVP was also treated by multi-derivative methods by E. H. Twizell and S. I. A. Tiraizi [19].

The main objective of the present paper is to apply a non-polynomial quintic spline function [15, 16, 17] that has a polynomial and trigonometric parts to develop a new numerical method for obtaining smooth approximations to the solution of nonlinear fourth-order DEs of the form (1) subject to (2). The technique, being reported, connects spline values at mid knots and their corresponding values of the fourth-order derivatives. New algorithms are constructed and their pertaining approximate solutions are compared with the solutions obtained by E. H. Twizell [18]. The paper is organized as follows. In Section 2, we give a description of the fourth-order DEs modelling some physical problems. Section 3 contains a brief introduction to the use of nonpolynomial quintic splines, where the spline relations to be used for discretization of the system (1) are presented. This topic is further elaborated on in Section 4. In Section 5, we present our numerical method for a system of nonlinear fourth-order BVPs and evaluate their truncation error. in Section 6, numerical results are provided to compare and demonstrate the efficiency of these methods. These results demonstrate that our algorithm performs better than other collocation, finite difference and spline methods. Section 7 concludes the paper with some remarks.

2. Description of the Problem

Fourth-order DEs occur in various physical problems which include certain phenomena related to the theory of elastic stability. The following classical fourth-order DE

$$EI\frac{d^4u}{dx^4} + P\frac{d^2u}{dx^2} = q, \tag{13}$$

arises in the beam-column theory [21]. Here u is the lateral deflection, q is the intensity of a distributed lateral load, P is the axial compressive force applied to the beam and EI represents the flexural rigidity in the plane of bending. Various generalizations of this equation which describes the deformation of an elastic beam with different types of two-point BCs have in general been extensively studied during the last two decades via a broad range of methods. In particular C. P. Gupta [22, 23] studied the DE of the form

$$\tfrac{d^4u}{dx^4} + g(x,u,u',u'') = e(x), \qquad x \in (0,1),$$

and more generally the equation

$$\frac{d^4u}{dx^4} + f(u)u' + g(x,u,u',u'') = e(x), \qquad x \in (0,1), \tag{14}$$

where f is a continuous function, g is a Caratheodory function satisfying the inequalities

$$g(x,u,v,w) \geq a(x)u^2 + b(x)|uv| + c(x)|uw| + d(x)|u|,$$

$$|g(x,u,v,w)| \leq |\alpha(x,u,v)|\,|w|^2 + \beta(x),$$

with real-valued functions $a(x), b(x), c(x), d(x), \alpha(x,u,v)$ and $\beta(x)$. The main tool used by Gupta is the Leray-Schauder continuation theorem. In the same direction Grossinho and Tersian [24] considered the BVP

$$\begin{cases} u^{(4)}(x) + g(u(x)) = 0, & x \in (0,1), \\ u''(0) = -f\big(-u'(0)\big), \quad u'''(0) = -h(u(0)), & u''(1) = u'''(1) = 0, \end{cases}$$

where g is a strictly monotone function that have some discontinuous f and h, which are unbounded , continuous and strictly increasing functions, defined on finite open intervals. The existence of pertaining solutions is treated through a dual variational method [25].

Finally, we mention the BVP

$$\begin{cases} u^{(4)}(x) = f(x,u(x)), & x \in (0,1), \\ u(0) = u'(0) = u''(1) = 0, & u'''(1) = g(u(1)), \end{cases}$$

that has been studied in the works of T. F. Ma [26] and Ma and Silva [27]. Existence of its solutions was proved in [26] using the mountain pass theorem. Uniqueness of these solutions was investigated in [27], under more restrictive assumptions on the function f, by means of the fixed point theorem for contractive mappings.

3. Nonpolynomial Quintic Spline

A quintic spline function $S_\triangle(x)$, interpolating a function $u(x)$ defined on $[a,b]$, is such that:

(i) In each subinterval $[x_{j-1}, x_j]$, $S_\triangle(x)$ is a polynomial of degree at most five.

(ii) The first, second, third and fourth derivatives of $S_{\triangle}(x)$ are continuous on $[a,b]$.

To be able to deal effectively with the posing BVPs, we introduce ‘spline functions’ containing a parameter τ. These are ‘non-polynomial splines’ defined through the solution of a differential equation in each subinterval. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. These ‘splines’ belong in the class C^2 and reduce into polynomial splines as the parameter $\tau \to 0$. The exact form of the spline depends upon the manner in which the parameter is introduced [28, 29]. The considered parametric spline functions can be: a spline under compression, spline under tension and an adaptive spline. A number of associated spline relations have been obtained for subsequent use.

A function $S_{\triangle}(x,\tau)$ of class $C^4[a,b]$ which interpolate $u(x)$ at the mesh points $\{x_j\}$ depends on a parameter τ, reduces to an ordinary quintic spline $S_{\triangle}(x)$ in $[a,b]$ as $\tau \to 0$ is termed as a parametric quintic spline function. The three parametric quintic splines are derived from a usual quintic spline by introducing the parameter in three different ways. The outcomes are termed as ‘parametric quintic spline-I’, ‘parametric quintic spline-II’ and ‘adaptive quintic spline’.

The spline function we propose in this paper is either span $\{1,x,x^2,x^3,\sin|\tau|x,\cos|\tau|x\}$, or span$\{1,x,x^2,x^3,\sinh|\tau|x,\cosh|\tau|x\}$, or the span $\{1,x,x^2,x^3,x^4,x^5\}$, when $\tau = 0$. The fact before, related to $\tau = 0$, is evident when the correlation between polynomial and non-polynomial spline bases are conceived in the following manner:

$$T_5 = \text{span } \{1,x,x^2,x^3,\sin(\tau x),\cos(\tau x)\},$$
$$= \text{span } \left\{1,x,x^2,x^3, \tfrac{24}{\tau^4}[\cos(\tau x) - 1 + \tfrac{(\tau x)^2}{2}],\ \tfrac{120}{\tau^5}[\sin(\tau x) - (\tau x) + \tfrac{(\tau x)}{6}^3]\right\}.$$

Clearly $\lim\limits_{\tau\to 0} T_5 = \{1,x,x^2,x^3,x^4,x^5\}$, where τ is the frequency of the trigonometric part of the spline function which can be real or pure imaginary.

This approach has the advantage over finite difference methods that it provides continuous approximation not only to $y(x)$, but also to y', y'' and higher-order derivatives at every point of the range of integration. Also, the C^{∞} –differentiability of the trigonometric part of non-polynomial splines compensates for the loss of smoothness inherent in polynomial splines.

4. Development of the Main Recurrence Relation

Our numerical method for approximating the solution to a differential equation of the type (1) deals, without loss of generality, with a uniform mesh $\triangle$ with nodal points x_i on $[a,b]$ such that

$$\triangle \quad : a = x_0 < x_1 < x_2 < x_3 \ldots\ldots\ldots < x_N = b$$
$$x_i = a + ih,\ i = 0,1,2,\ldots..N,\ and\ \ h = \tfrac{b-a}{N}.$$

Consider then a non-polynomial function $S_{\triangle}(x)$ of the class $C^4[a,b]$, dependent on a parameter τ, which interpolates $y(x)$ at the mesh points x_i, $i = 0,1,2,\ldots..N$, and reduces to an ordinary quintic spline $S_{\triangle}(x)$ in $[a,b]$ as $\tau \to 0$.

For each segment $[x_i,\ \ x_{i+1}]$, $i = 0,1,2,\ldots..N-1$, the non-polynomial $S_{\triangle}(x)$ is defined by

$$S_{\triangle}(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 + e_i\ sin\tau(x-x_i) + f_i\ cos\tau(x-x_i),$$
$$i = 0,1,2,\ldots..N-1, \tag{15}$$

where a_i, b_i, c_i, d_i, e_i and f_i are constants.
Let y_i be an approximation to $y(x_i)$, obtained by the segment $S_\triangle(x)$ of the mixed splines function passing through the points $(x_i,\ \ y_i)$ and $(x_{i+1},\ \ y_{i+1})$. To satisfy the necessary conditions for the coefficients introduced in (15), we do not only require that $S_\triangle(x)$ satisfies interpolatory conditions at x_i and x_{i+1} , but also the continuity of first, second and third derivatives at the common nodes $(x_i,\ y_i)$.
To derive an expression for the coefficients of (15) in terms of $y_i, y_{i+1}, D_i, D_{i+1}, F_i$ and F_{i+1} , we need the following notation.

$$
\begin{aligned}
&S_\triangle(x_i) = y_i, \qquad S_\triangle(x_{i+1}) = y_{i+1},\\
&S'_\triangle(x_i) = D_i, \qquad S'_\triangle(x_{i+1}) = D_{i+1},\\
&S^{(4)}_\triangle(x_i) = F_i, \quad S^{(4)}_\triangle(x_{i+1}) = F_{i+1}.
\end{aligned}
\tag{16}
$$

Some algebraic manipulations lead then to

$$
\begin{aligned}
&a_i = y_i - \frac{F_i}{\tau^4}\ ,\\
&b_i = D_i - \frac{F_{i+1} - F_i\ Cos\,\theta}{\tau^3 Sin\,\theta}\ ,\\
&c_i = -\frac{2D_i + D_{i+1}}{h} + \frac{3(y_{i+1} - y_i)}{h^2} - \frac{3(F_{i+1} - F_i)}{\tau^4 h^2} + \frac{\left[S_{i+1}\left(2 + Cos\,\theta\right) - S_i(1 + 2\,Cos\,\theta)\right]}{h\tau^3 Sin\,\theta},\\
&d_i = \frac{D_{i+1} + D_i}{h^2} + \frac{2(y_i - y_{i+1})}{h^3} - \frac{(F_{i+1} - F_i)(1 + Cos\,\theta)}{\tau^3 h^2 Sin\,\theta} + \frac{2(F_{i+1} - F_i)}{\tau^4 h^3}\ ,\\
&e_i = \frac{F_{i+1} - F_i Cos\theta}{\tau^4 Sin\theta}\ ,\\
&f_i = \frac{F_i}{\tau^4}\ ,
\end{aligned}
\tag{17}
$$

where $\theta = \tau h$ and $i = 0,1,2\ldots\ldots\ldots\ldots.N-1.$
The posing differential equation

$$y^{(4)}(x) = f(x,y), \qquad a < x < b, \quad a,b,x \in R$$

can be considered in nonlinear form as

$$y^{(4)} + f(x,y)\ q(y) = g(x), \qquad\qquad x \in [a,b]. \tag{18}$$

Furthermore, when subjected to BCs (2), this DE can be discretized viz

$$F_i + f_i\, q_i\ = g_i, \qquad 0 < x < 1,$$

where $F_i = S''''_\triangle(x_i)$, $g_i = g(x_i)$, $f_i = f(x_i,y_i)$ and $q_i = q(y_i)$.
Using the continuity of the first and third-order derivatives at $(x_i,\ \ y_i)$, that is $S'_{\triangle-i}(x_i) = S'_{\triangle i}(x_i)$ and $S'''_{\triangle-i}(x_i) = S'''_{\triangle i}(x_i)$, we obtain for $i = 1,2,\ldots,N$ the following relations,

$$D_{i-1}+4D_i+D_{i+1} = \tfrac{6}{h^2}(y_{i-1}-2yF_i+y_{i+1})+\tfrac{6h^2}{\theta^2}\left(\tfrac{1}{\theta\,Sin\,\theta}-\tfrac{1}{\theta^2}-\tfrac{1}{6}\right)(F_{i-1}+F_{i+1})$$
$$+\tfrac{6h^2}{\theta^2}\left(\tfrac{2}{\theta^2}-\tfrac{2\,Cos\,\theta}{\theta Sin\theta}-\tfrac{4}{6}\right)F_i,\qquad i=1,2,\ldots..,N. \tag{19}$$

and

$$D_{i-1}-2D_i+D_{i+1} = h^2\left(\tfrac{1}{\theta\,Sin\,\theta}-\tfrac{1}{\theta^2}\right)(F_{i-1}+F_{i+1})$$
$$+2h^2\left(\tfrac{1}{\theta^2}-\tfrac{Cos\,\theta}{\theta Sin\theta}\right)F_i,\qquad i=1,2,\ldots..,N. \tag{20}$$

By adding equations (19)and (20), we obtain

$$D_i = \tfrac{1}{h^2}(y_{i-1}-2y_i+y_{i+1})+h^2\left(\tfrac{1}{\theta^3 Sin\theta}-\tfrac{1}{6\theta\,Sin\,\theta}-\tfrac{1}{\theta^4}\right)(F_{i-1}+F_{i+1})$$
$$+h^2\left(\tfrac{2}{\theta^4}-\tfrac{2\,Cos\,\theta}{\theta^3 Sin\,\theta}+\tfrac{2\,Cos\,\theta}{6\,\theta\,Sin\theta}-\tfrac{1}{\theta^2}\right)F_i,\qquad i=1,2,\ldots..,N. \tag{21}$$

Then from Equations (20)and (21) we get

$$y_{i+2}-4y_{i+1}+6\,y_i-4y_{i-1}+y_{i-2} = h^4[\alpha(F_{i+2}+F_{i-2})+\beta(F_{i+1}+F_{i-1})+\gamma F_i],$$
$$i=1,2,\ldots..,N, \tag{22}$$

where $F_i = -f_iq_i+g_i$, $f_i = f(x_i,y_i)$, $q_i = q(y_i)$, and $g_i = g(x_i)$, along with

$$\alpha = \left(\frac{-1}{\theta^3 Sin\,\theta}+\frac{1}{6\theta\,Sin\theta}+\frac{1}{\theta^4}\right),\quad \beta = \left(\frac{2(1+Cos\,\theta)}{\theta^3 Sin\,\theta}-\frac{Cos\,\theta-2}{3\theta\,Sin\theta}-\frac{4}{\theta^4}\right),$$

and

$$\gamma = \left(-\frac{2(1+2\,Cos\theta)}{\theta^3 Sin\,\theta}-\frac{4\,Cos\,\theta-1}{3\theta\,Sin\theta}+\frac{6}{\theta^4}\right).$$

5. Description of the Method

At the mesh point x_i the proposed nonlinear DE (18)

$$y^{(4)}+f(x,y)\,q(y) = g(x),\qquad x\in[a,b],$$

subjected to BCs (2), may be discretized as before viz

$$F_i+f_i\,q_i = g_i,\qquad 0<x<1.$$

The associated equation (22) gives $(N-2)$ nonlinear equations in the N unknowns y_i, $i = 1,2,\ldots,N$. Obviously two more equations are needed. One at each end of the range of integration related to the direct decomposition of y_i. These two end conditions can be derived as follows.

Equation (19) for $i = 1,2,$ rewrites as

$$D_0+4D_1+D_2 = \frac{6}{h^2}(y_0-2y_1+y_2)+\alpha_1(F_0+F_2)+\beta_1F_1, \tag{23}$$

and

$$D_1 + 4D_2 + D_3 = \frac{6}{h^2}(y_1 - 2y_2 + y_3) + \alpha_1(F_1 + F_3) + \beta_1 F_2, \tag{24}$$

We also write (20) for $i = 1, 2,$ as

$$D_0 - 2D_1 + D_2 = \alpha_2(F_0 + F_2) + \beta_2 F_1, \tag{25}$$

and

$$D_1 - 2D_2 + D_3 = \alpha_2(F_1 + F_3) + \beta_2 F_2, \tag{26}$$

where

$$\alpha_1 = \frac{6h^2}{\theta^2}\left(\frac{1}{\theta\, Sin\,\theta} - \frac{1}{\theta^2} - \frac{1}{6}\right), \quad \beta_1 = \frac{6h^2}{\theta^2}\left(\frac{2}{\theta^2} - \frac{2\,Cos\theta}{\theta\, Sin\,\theta} - \frac{4}{6}\right),$$

$$\alpha_2 = h^2\left(\frac{1}{\theta\, Sin\,\theta} - \frac{1}{\theta^2}\right), \quad \beta_2 = 2h^2\left(\frac{1}{\theta^2} - \frac{Cos\,\theta}{\theta\, Sin\,\theta}\right).$$

Subtracting, on one hand, equation (25) from (23) leads to

$$D_1 = \frac{1}{h^2}(y_0 - 2y_1 + y_2) + \frac{\alpha_1 - \alpha_2}{6}(F_0 + F_2) + \frac{\beta_1 - \beta_2}{6} F_1. \tag{27}$$

On the other hand, subtraction of equqtion (26) from (24) yields

$$D_2 = \frac{1}{h^2}(y_1 - 2y_2 + y_3) + \frac{\alpha_1 - \alpha_2}{6}(F_1 + F_3) + \frac{\beta_1 - \beta_2}{6} F_2. \tag{28}$$

Substitute then equations (27) and (28) into (23) to obtain the first end condition

$$-2y_0 + 5y_1 - 4y_2 + y_3 = -h^2 D_0 + h^4(\omega_0 F_0 + \omega_1 F_1 + \omega_2 F_2 + \omega_3 F_3); \qquad i = 1, \tag{29}$$

where

$$\omega_0 = \left(\frac{2}{\theta^3 Sin\,\theta} + \frac{1}{6\theta\, Sin\theta} - \frac{2}{\theta^4} - \frac{1}{2\theta^2}\right), \quad \omega_1 = \left(\frac{1-2Cos\,\theta}{6\theta\, Sin\theta} - \frac{1+4Cos\theta}{\theta^3 Sin\theta} + \frac{5}{\theta^4} - \frac{1}{\theta^2}\right),$$

$$\omega_2 = \left(\frac{1-2Cos\,\theta}{6\theta\, Sin\theta} + \frac{1}{\theta^3 Sin\theta} - \frac{1}{\theta^4}\right), \quad \omega_3 = \left(\frac{1}{6\theta\, Sin\theta} + \frac{1}{\theta^3 Sin\theta} - \frac{1}{\theta^4} - \frac{1}{3\theta^2}\right).$$

Similarly, we establish the second end condition

$$y_{N-2} - 4\,y_{N-1} + 5\,y_N - 2\,y_{N+1} = -h^2 D_{N+1} + h^4(\omega_3 F_{N-2} + \omega_2 F_{N-1} + \omega_1 F_N + \omega_0 F_{N+1}); \quad i = N. \tag{30}$$

The local truncation errors t_i, $i = 1, 2, \ldots\ldots, N$, associated with the conditions (22),(29) and (30) can be determined as follows. Rewrite the conditions (22), (29) and (30) in the form:

$$-2y_0 + 5y_1 - 4y_2 + y_3 = -h^2 y_0^{(2)} + h^4\left(\omega_0 y_0^{(4)} + \omega_1 y_1^{(4)} + \omega_2 y_2^{(4)} + \omega_3 y_3^{(4)}\right) + t_i; \quad i = 1, \tag{31}$$

$$y_{i+2} - 4y_{i+1} + 6\,y_i - 4y_{i-1} + y_{i-2} = h^4\Big[\alpha\Big(y_{i+2}^{(4)} + y_{i-2}^{(4)}\Big) + \beta\Big(y_{i+1}^{(4)} + y_{i-1}^{(4)}\Big) + \gamma y_i^{(4)}\Big] + t_i;$$
$$i = 2, 3, \ldots\ldots\ldots, N-1, \tag{32}$$

and

$$y_{N-2} - 4\,y_{N-1} + 5\,y_N - 2\,y_{N+1} = -h^2 y_{N+1}^{(2)}$$
$$+ h^4\Big(\omega_3 y_{N-2}^{(4)} + \omega_2 y_{N-1}^{(4)} + \omega_1 y_N^{(4)} + \omega_0 y_{N+1}^{(4)}\Big) + t_N; \quad i = N. \tag{33}$$

The terms y_{i-2} and $y_{i+2}^{(4)}$,.... in (32) are expanded around the point x_i using Taylor series. Consequently, the expressions for $t_i,\ i = 2, 3, \ldots\ldots, N-1,$ are

$$t_i = (1 - 2\alpha - 2\beta - \gamma)\, h^4 y_i^{(4)} + \Big(\tfrac{1}{6} - 4\alpha - \beta\Big)\, h^6 y_i^{(6)}$$
$$+ \Big(\tfrac{1}{80} - \tfrac{16}{12}\alpha - \tfrac{1}{12}\beta\Big) h^8 y_i^{(8)} + \Big(\tfrac{17}{30240} - \tfrac{8}{45}\alpha - \tfrac{1}{360}\beta\Big) h^{10} y_i^{(10)} + O(h^{11}),$$
$$i = 2, 3, \ldots\ldots, N-1. \tag{34}$$

Also expressions for $t_i,\ i = 1, N,$ are obtained by expanding equations (31) and (33), using Taylor series, around the point x_0 and x_N, respectively, viz

$$t_i = (\tfrac{22}{24} - \omega_0 - \omega_1 - \omega_2 - \omega_3) h^4 y_i^{(4)} + (1 - \omega_1 - 2\omega_2 - 3\omega_3) h^5 y_i^{(5)}$$
$$+ \Big(\tfrac{478}{720} - \tfrac{1}{2}\omega_1 - 2\omega_2 - \tfrac{9}{2}\omega_3\Big) h^6 y_i^{(6)} + \Big(\tfrac{1680}{5040} - \tfrac{1}{6}\omega_1 - \tfrac{4}{3}\omega_2 - \tfrac{9}{2}\omega_3\Big) h^7 y_i^{\{7\}}$$
$$+ \Big(\tfrac{5542}{40{,}320} - \tfrac{1}{24}\omega_1 - \tfrac{2}{3}\omega_2 - \tfrac{27}{8}\omega_3\Big) h^8 y_i^{(8)} + O(h^9), \qquad i = 1, N. \tag{35}$$

For any choice of α, β and γ, provided that $\gamma = 1 - 2\,\alpha - 2\beta,$ the conditions (22), (29) and (30) give rise to a family of methods of different orders, which may be labelled as follows.
Case I : Second-order method
For which

$$\alpha = \tfrac{-5}{3559},\ \beta = \tfrac{3679}{21{,}354},\ \gamma = \tfrac{14056}{21{,}354},\ \omega_0 = \tfrac{6}{45},\ \omega_1 = \tfrac{41}{72},\ \omega_2 = \tfrac{19}{90} \text{ and } \omega_3 = \tfrac{1}{360}.$$

Case II : Fourth-order method
For which

$$\alpha = \tfrac{-10}{7199},\ \beta = \tfrac{7439}{43{,}194},\ \gamma = \tfrac{28{,}436}{43{,}194},\ \omega_0 = \tfrac{7}{90},\ \omega_1 = \tfrac{49}{72},\ \omega_2 = \tfrac{7}{45} \text{ and } \omega_3 = \tfrac{1}{360}.$$

Clearly, our family of numerical methods is represented by the discretized equations (22), the BCs and by the solution vector $Y = [y_1, y_2, \ldots\ldots, y_N]^T$, T denoting transpose. Y is obtained by solving a non-linear algebraic system of order N [30]. To ensure cost effectiveness, better accuracy and applicable simplicity of the new method, one needs to fix first the unknown parameters α, β and γ, which are the expressions containing the actual parameter τ. This new approach is remarkable in producing a family of fourth- and second-order methods by running the code only once, while saving on the multiplications involved in the α, β and γ expressions. The computational cost of the new approach is same as of methods based on standard polynomial spline functions [12], whereas its accuracy is two-fold better.

6. Numerical Experiments

We now consider two numerical examples illustrating the comparative performance of nonpolynomial quintic spline (NPQS) algorithms with the fourth-order convergent finite difference method of [18].

Example 6.1. Consider the BVP

$$y^{(4)}(x) = 6\exp[-4y(x)] - 12(1+x)^{-4}, \qquad 0 < x < 1, \tag{36}$$

with the BCs

$$y(0) = 0, \quad y(1) = \ln 2, \quad y^{(2)}(0) = 1, \; y^{(2)}(1) = -0.25. \tag{37}$$

The analytic solution of this BVP is

$$y(x) = \ln(1+x). \tag{38}$$

Example 6.2. Solve the same DE of the previous example, subject to the BCs

$$y(0) = 0, \; y(1) = \ln 2, \; y^{(1)}(0) = 1, \; y^{(1)}(1) = 0.5. \tag{39}$$

The analytic solution of this BVP happens to be identical to the solution (38) of the preceeding BVP.

6.1. **Nonpolynomial quintic spline solutions**

The maximum absolute error of our NPQS algorithms and of the finite difference method of Twizell [18] for Examples 6.1 and 6.2 are listed in Tables 1 and 2 respectively.

Table 1. Comparison between errors of solutions to Example 6.1

	Finite Difference Method	Second-order NPQS	Fourth-order NPQS
N	(Maximum Absolute Error)	(Maximum Absolute Error)	(Maximum Absolute Error)
8	0.37E-5	0.63E-6	0.27E-7
16	0.29E-6	1.27E-7	2.11E-9
32	0.19E-7	3.71E-8	1.83E-10

Table 2. Comparison between errors of solutions to Example 6.2

	Finite Difference Method	Second-order NPQS	Fourth-order NPQS
N	(Maximum Absolute Error)	(Maximum Absolute Error)	(Maximum Absolute Error)
8	0.22E-4	0.39E-6	0.12E-6
16	0.42E-5	1.12E-7	9.17E-9
32	0.67E-6	0.32E-7	6.23E-10

Acknowledgements
The authors are grateful to an anonymous referee for a number of useful comments.

7. Conclusion

In this paper we used a nonpolynomial quintic spline function to develop numerical algorithms for the solution of nonlinear fourth-order boundary value problems. The reported approach generalizes usual nonpolynomial spline algorithms and provides a solution at every point of the range of integration. The numerical results, illustrated in Tables 1 and 2, demonstrate that our algorithm performs better than the fourth-order convergent finite difference method.

References

[1] M. A. Ramadan, I. F. Lashien, and W. K. Zahra, Quintic nonpolynomial spline solutions for fourth order two-point boundary value problem, *Communications in Nonlinear Science and Numerical Simulation* **14** (2008), 1105-1114.

[2] Siraj-ul-Islam, I. A. Tirmizi, F. Haq, and S. K. Taseer, Family of numerical methods based on non-polynomial splines for solution of contact problems, *Communications in Nonlinear science and Numerical Simulation* **13** (2008), 1448-1460.

[3] S. S. Siddiqi and G. Akram, Solution of the system of fourth-order boundary value problems using non-polynomial spline technique, *Applied Mathematics and Computation* **185** (2007), 128-135.

[4] Siraj-ul-Islam, I. A. Tirmizi, and S. Ashraf, A class of methods based on non- polynomial spline functions for the solution of a special fourth-order boundary value problems with engineering applications, *Applied Mathematics and Computation* **174** (2006), 1169-1180.

[5] J. Rashidinia, R. Jalilian, and R. Mohammadi, Comment on the paper : A class of methods based on non-polynomial spline functions for the solution of a special fourth- order boundary value problems with engineering applications, *Applied Mathematics and Computation* **186** (2007), 1572-1580.

[6] E. A. Al-Said, M. Aslam Noor, and T. M. Rassias, Cubic splines method for solving fourth-order obstacle problems, *Applied Mathematics and Computation* **174** (2006), 180-187.

[7] E. A. Al-Said, and M. Aslam Noor, Quartic spline method for solving fourth order obstacle boundary value problems, *Journal of Computational and Applied Mathematics* **143** (2002), 107-116.

[8] R. A. Usmani, Finite difference methods for a certain two point boundary value problem, *Indian Journal of Pure and Applied Mathematics* **14** (3) (1983), 398-411.

[9] R. A. Usmani, Smooth spline approximations for the solution of a boundary value problem with engineering applications, *Journal of Computational and Applied Mathematics* **6** (1980), 93-98.

[10] G. B. Loghmani, and S. R. Alavizadeh, Numerical solution of fourth-order problems with separated boundary conditions, *Applied Mathematics and Computation* **191** (2007), 571- 581.

[11] J. Rashidinia, and A. Golbabaee, Convergence of numerical solution of a fourth-order boundary value problem. *Applied Mathematics and Computation* **171** (2005), 1296-1305.

[12] S. S. Siddiqi, and G. Akram, *Quintic Spline Solutions of Fourth Order Boundary Value Problems*, Math. NA/0306357, 2003.

[13] M. Van Daele, G. Vanden Berghe, and H. A. De Meyer, Smooth approximation for the solution of a fourth order boundary value problem based on nonpolynomial splines, *Journal of Computational and Applied Mathematics* **51**(1994), 383-394.

[14] R. A. Usmani. The use of Quartic splines in the numerical solution of a fourth-order boundary value problems, *Journal of Computational and Applied Mathematics* **44** (1992), 187-199.

[15] M. Kumar, and P. K. Srivastava, Computational techniques for solving differential equations by quadratic, quartic and octic Spline, *Advances in Engineering Software* **39** (2008), 646-653.

[16] M. Kumar, and P. K. Srivastava, Computational techniques for solving differential equations by cubic, quintic and sextic spline, *International Journal for Computational Methods in Engineering Science and Mechanics* **10** (1) (2009), 108-115.

[17] J. Rashidinia, R. Jalilian, and R. Mohammadi, Non-polynomial spline methods for the solution of a system of obstacle problems, *Applied Mathematics and Computation* **188** (2) (2007), 1984-1990.

[18] E. H. Twizell, *A Two-Grid, Fourth Order Method for Nonlinear Fourth Order Boundary Value Problems*, Technical Report TR/12/85, Brunel University Department of Mathematics and Statistics, 1985.

[19] E. H. Twizell, and S. I. A. Tiraizi, *Multiderivative Methods for Non-Linear Fourth Order Boundary Value Problems*, Technical Report TR/07/85, Brunel University Department of Mathematics and Statistics, 1985.

[20] R. P. Agarwal, and G. Akrivis, Boundary value problem occurring in plate deflection theory, *Journal of Computational and Applied Mathematics* **8** (1982), 145-154.

[21] S. Timoshenko, and J. Gere, *Theory of Elastic stability*, *2nd Edition*, McGraw Hill, New York, 1961.

[22] C. P. Gupta, Existence and uniqueness theorems for the bending of an elastic beam equation, *Applicable Analysis* **26** (1988), 289-304.

[23] C. P. Gupta, Existence and uniqueness theorems for a fourth order boundary value problem of Sturm-Liouville type, *Differential and Integral Equations* **4** (2) (1991), 397-410.

[24] M. R. Grossinho, and S. A.Tersian, The dual variational principal and equilibria for a beam resting on a discontinuous nonlinear elastic foundation, *Nonlinear Analysis ATM* **41** (3-4) (2000), 417-431.

[25] A. Ambrosettoi, and M. Badiale, The dual variational principle and elliptic problems with discontinuous nonlinearities, *Journal of Mathematical Analysis and its Applications* **140** (1989), 363-373.

[26] T. F. Ma, Positive solutions for a beam equation on a nonlinear elastic foundation, *Mathematical and Computational Modelling* **39** (2004), 1195-1201.

[27] T. F. Ma, and J. da Silva, Iterative solutions for a beam equation with nonlinear boundary conditions of third order, *Applied Mathematics and Computation* **159** (2004), 11-18.

[28] P. K. Srivastava, and M. Kumar, Numerical treatment of nonlinear third order boundary value problem, *Applied Mathematics* **2** (2011), 959-964.

[29] P. K. Srivastava, M. Kumar, and R. N. Mohapatra, Quintic nonpolynomial spline method for the solution of a second-order boundary-value problem with engineering applications, *Computers and Mathematics with Applications* **62** (2011), 1707-1714.

[30] Siraj-ul-Islam, I. A. Tirmizi, F. Haq, and M. A. Khan, Non-polynomial splines approach to the solution of sixth-order boundary-value problems, *Applied Mathematics and Computation* **195** (2008), 270-284.

Article history: Submitted December, 28, 2011; Accepted March, 03, 2012.