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Perturbation Estimates for the Two Kinds of Algebraic Riccati Equations Arising in a Stochastic Control

Dedicated to professor Milko Petkov on the occasion of his 80th birthday.

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Abstract. *This paper is concerned with two kinds algebraic Riccati equations arising in the linear quadratic (LQ) control. The problem gives rise to the existence of a so-called stabilizing solution to a stochastic (generalized) algebraic Riccati equation, which is however fundamentally different from the classical algebraic Riccati equation as a result from the extension of the LQ problem.*
Perturbation estimates for the stabilizing solution of the stochastic algebraic Riccati equations are derived. The new perturbation results are illustrated by numerical examples.

Key words : Stochastic LQ Control, Generalized Algebraic Riccati Equation, Stabilizing Solution, Perturbation Bound.

AMS Subject Classifications : 15A24, 15A45, 65F35

1. Introduction

In this paper we consider the perturbation theory for the two kinds of algebraic Riccati equations. The first is the following continuous time Riccati equation:

$$A^*X + XA - XGX + Q + S^*XS = 0, \tag{1}$$

where $A, S \in \mathbb{C}^{m\times m}$ and $Q, G \in \mathbb{H}^{m\times m}$, G is a positive definite matrix and Q is a positive semidefinite matrix. Here $\mathbb{C}^{m\times m}$ denotes the set of $m \times m$ complex matrices and $\mathbb{H}^{m\times m}$ is the set of $m \times m$ Hermitian complex matrices.
The second kind is the following discrete time Riccati equation:

$$X = \mathcal{P}(X) := C^*XC - C^*XB\,(R + B^*XB)^{-1}\,(C^*XB)^* + Q + S^*XS, \tag{2}$$

where $C, B, S \in \mathbb{C}^{m\times m}$ and $R, Q \in \mathbb{H}^{m\times m}$, R and Q are positive definite.

Many authors have studied the more general Riccati equation

$$\mathcal{R}(X) \;:=\; A^*X + XA - XGX + Q + \Sigma(X) = 0 \;, \tag{3}$$

where $\Sigma(X)$ stands for a linear operator in the positive definite sense, and $\Sigma(X) = S^*XS$ in the case of (1). The stabilizing solutions to nonlinear equations (1) and (2) are very important to applications. There are many papers where the properties and algorithms for finding the stabilizing solutions are studied (see the reference list). In order to ensure the existence of a unique stabilizing solution of considered equations it is not sufficient to assume the well known assumptions in the case of classical linear quadratic control problems. It is necessary to consider more general concepts of stabilizability and detectability (for example see [6, 7]).

Linearly perturbed Riccati equations appear in stochastic control theory. The stochastic linear quadratic control problem has many applications in both theory and applications and it has been studied by many researches, for examples we refer to [1, 8, 19] and the literature there in. The stochastic linear quadratic control problems and stochastic Riccati equations arise in infinite time horizon where the diffusion term in dynamics depends on both the state and control variables. Applications lead us to the case, the control and state weighting matrices in the cost functional are indefinite, i.e. so-called indefinite stochastic linear quadratic problems have been investigated. Detailed discussion, explanations and examples can be found in [4]. In addition, comments and examples for portfolio selection in finance through stochastic quadratic models occur in investigations of [5, 20, 21]. The analytical and computational approaches, based on linear matrix inequalities, to solving more complicated Riccati equations with possibly indefinite cost matrices have been studied in [11, 12, 14, 16].

In this paper perturbation bounds for the stabilizing solution of the above generalized algebraic Riccati equations are derived. We denote by $\mathbb{C}_-$ the set of complex numbers with negative real parts. The spectrum of any complex matrix A will be denoted by $\sigma(A)$. A matrix A is said to be c-stable if all the eigenvalues of A lie in the open left half plane, i.e. $\sigma(A) \in \mathbb{C}_-$. For a linear operator $\mathcal{L}$ on $\mathbb{H}^{m\times m}$, let $\rho(\mathcal{L}) = \max\{|\lambda|: \lambda \in \sigma(\mathcal{L})\}$ be the spectral radius. $\mathcal{L}$ is called c-stable if the eigenvalues to $\mathcal{L}$ lie in the open left half plane, i.e. $\sigma(\mathcal{L}) \in \mathbb{C}_-$. A matrix A is said to be d-stable if all the eigenvalues of A lie in the open unit disk, i.e. $\sigma(A) < 1$. A linear operator $\mathcal{L}$ on $\mathbb{H}^{m\times m}$ is called d-stable if the eigenvalues to $\mathcal{L}$ lie in the open unit disk, i.e. $\sigma(\mathcal{L}) < 1$.

The symbol $\|\,..\,\|$ stands for any unitary invariant matrix norm, $\|\,..\,\|_F$ is the Frobenius norm, and $\|\,..\,\|_2$ is the spectral norm. We will often use the norm inequality $\|AB\| \le \|A\|_2\,\|B\|$.

2. A Perturbation Bound for the Continuous-Time Case

Equation (1) is considered as a special case of (3). In this paper we will investigate how the stabilizing X of (1) is changed when the coefficient matrices to (1) are slightly perturbed and the new perturbation equation is obtained:

$$\tilde{A}^*\tilde{X} + \tilde{X}\tilde{A} - \tilde{X}\tilde{G}\tilde{X} + \tilde{Q} + \tilde{S}^*\tilde{X}\tilde{S} = 0\,, \tag{4}$$

where $\tilde{A} = A + \Delta A$, $\tilde{G} = G + \Delta G$, $\tilde{Q} = Q + \Delta Q$, $\tilde{S} = S + \Delta S$, $\tilde{X} = X + \Delta X$, with ΔQ, ΔG, $\Delta X \in \mathbb{H}^{m\times m}$ and ΔA, $\Delta S \in \mathbb{C}^{m\times m}$.

We will derive a perturbation bound for the stabilizing solution X to (1). Assuming that $\tilde{X}$ approximates the stabilizing solution X we are interested first, to derive sufficient conditions the matrix $\tilde{X}$ to be the stabilizing solution to new perturbation equation (4), and second, to estimate $\|X - \tilde{X}\|$.

2.1. **Preliminaries**

For a given $\Phi \in \mathbb{C}^{m\times m}$ we define the Lyapunov operator $\mathbf{L}_\Phi$ by:

$$\mathbf{L}_\Phi W = \Phi^* W + \Phi W, \quad W \in \mathbb{H}^{m\times m}. \tag{5}$$

The Lyapunov operator is most important for investigations in the field of control theory [2, 7]. It is well known that if Φ is a c-stable matrix then $\mathbf{L}_\Phi$ is an invertible operator. We begin with the following lemma derived by Sun.

Lemma [17], [8] **2.1**. *Let* $\mathbf{L}_\Phi$ *be the linear operator defined by (5) with a c-stable matrix* $\Phi \in \mathbb{C}^{m\times m}$. *If* $E \in \mathbb{C}^{m\times m}$ *satisfies*

$$2\|\mathbf{L}_\Phi^{-1}\|\,\|E\| < 1,$$

then $\Phi + E$ *is a c-stable matrix.*

It is easy to conclude that if a matrix Φ is c-stable then the Lyapunov operator $\mathbf{L}_\Phi$ is resolvent positive (see section 3 of [2]and section 3of [7]).

Let us consider the algebraic Lyapunov equation:

$$\mathbf{L}_A X + \Pi_1(X) + Q = 0\,, \tag{6}$$

where $A, Q = Q^*$ are given $n \times n$ complex matrices and $\Pi_1(X)$ is a positive linear operator defined in $\mathbb{H}^{m\times m}$.

The next lemma generalizes Lyapunov's stability theorem.

Lemma [7] **2.2**. *For equation (6) the following statements are equivalent.*

(i) All eigenvalues of A lie in the open left half-plane and $\rho(L_A^{-1}\Pi_1) < 1$.

(v) $L_A + \Pi_1$ *is c-stable.*

If any one of these conditions is fulfilled then A is called c-stable relative to Π_1.

For the function $\mathcal{R}$, the first Fr'echet derivative of $\mathcal{R}$ at a matrix $X \in \mathbb{H}^{m\times m}$ is a linear map $\mathcal{R}'_X : \mathbb{H}^{m\times m} \to \mathbb{H}^{m\times m}$ given by

$$\mathcal{R}'_X(H) = (A - GX)^* H + H(A - GX) + \Sigma(H).$$

A matrix $X \in \mathbb{H}^{m\times m}$ is called stabilizing for $\mathcal{R}$ if $\sigma(\mathcal{R}'_X) \subset \mathcal{C}_-$. In addition, if a stabilizing matrix X is a solution to $\mathcal{R}(X) = 0$, then X is a stabilizing solution.

Remark 2.1. Note that $\mathcal{R}'_X = \mathbf{L}_{A-GX} + \Sigma$. So, if X is a stabilizing solution to $\mathcal{R}(X) = 0$ then $\mathcal{R}'_X$ is c-stable and thus $A - GX$ is c-stable relative to Σ. Thus all eigenvalues of $A - GX$ lie in the open left half-plane and $\rho(\mathbf{L}_{A-GX}^{-1}\Sigma) < 1$.

2.2. **A perturbation estimate**

In our investigation we carry out some matrix manipulations on the difference of the equations. Subtracting equation (1) from (4) we have

$$\begin{aligned}(A-GX)^*\Delta X+\Delta X(A-GX) = &-[\Delta Q+\Delta A^*X+X\Delta A-X\Delta GX]\\ &+S^*XS-\tilde{S}^*\tilde{X}\tilde{S}\\ &-[(\Delta A-\Delta GX)^*\Delta X+\Delta X(\Delta A-\Delta GX)]\\ &+\Delta X(G+\Delta G)\Delta X,\end{aligned}$$

which is the same as

$$\begin{aligned}(A-GX)^*\Delta X+\Delta X(A-GX) = &-[\Delta Q+\Delta A^*X+X\Delta A-X\Delta GX\\ &+S^*X\Delta S+\Delta S^*XS+\Delta S^*X\Delta S]-[(\Delta A-\Delta GX)^*\Delta X\\ &+\Delta X(\Delta A-\Delta GX)+\tilde{S}^*\Delta X\tilde{S}\big]+\Delta X(G+\Delta G)\Delta X.\end{aligned}$$

Hence

$$(A-GX)^*\Delta X+\Delta X(A-GX) = -E_1-E_2+h_1(\Delta X)+h_2(\Delta X)\,, \tag{7}$$

where

$$\begin{aligned}E_1 &= \Delta Q+\Delta A^*X+X\Delta A-X\Delta GX+S^*X\Delta S+\Delta S^*XS\,,\\ E_2 &= \Delta S^*X\Delta S\,,\\ h_1(\Delta X) &= -\big[(\Delta A-\Delta GX)^*\Delta X+\Delta X(\Delta A-\Delta GX)+\tilde{S}^*\Delta X\tilde{S}\big]\\ &= -[(\Delta A-\Delta GX)^*\Delta X+\Delta X(\Delta A-\Delta GX)+(S+\Delta S)^*\Delta X(S+\Delta S)]\\ h_2(\Delta X) &= \Delta X(G+\Delta G)\Delta X.\end{aligned}$$

Invoke the Lyapunov linear operator (5) for $\Phi = A_X = A-GX$:

$$\mathbf{L}_{A_X}W \ := \ A_X{}^*W+WA_X\,, \qquad W\in\mathbb{H}^{m\times m}\,.$$

Then

$$\mathbf{L}_{A_X}(\Delta X) = -E_1-E_2+h_1(\Delta X)+h_2(\Delta X)\,, \tag{8}$$

and we may introduce

$$\mathcal{L} = \mathcal{R}'_X = \mathbf{L}_{A_X}+\Sigma,$$

where $\Sigma(H) = S^*H\,S$ is a positive linear operator.

Furthermore, we define the following operators

$\mathbf{P}_Z \ : \ \mathbb{C}^{m\times m} \to \mathbb{H}^{m\times m}$ by

$\mathbf{P}_Z(N) \ := \ \mathbf{L}_{A_X}^{-1}(Z^*N + N^*Z)\,, \quad Z,N \in \mathbb{C}^{m\times m}\,,$

$\mathbf{Q} \ : \ \mathbb{H}^{m\times m} \to \mathbb{H}^{m\times m}$ by

$\mathbf{Q}(H) \ := \ \mathbf{L}_{A_X}^{-1}(XHX)\,, \quad H \in \mathbb{H}^{m\times m}\,,$

with

$$\|\mathbf{L}_{A_X}^{-1}\| \ = \ \max_{\substack{W \in \mathbb{H}^{m\times m} \\ \|W\| \neq 0}} \frac{\|\mathbf{L}_{A_X}^{-1}W\|}{\|W\|}\,, \quad \|\mathbf{Q}\| \ = \ \max_{\substack{W \in \mathbb{H}^{m\times m} \\ \|W\| \neq 0}} \frac{\|\mathbf{Q}W\|}{\|W\|}$$

and

$$\|\mathbf{P}_Z\| \ = \ \max_{\substack{Y \in \mathbb{C}^{m\times m} \\ \|Y\| \neq 0}} \frac{\|\mathrm{P}_Z Y\|}{\|Y\|}\,.$$

Here

$\|\mathbf{L}_{A_X}^{-1}W\| \leq \|\mathbf{L}_{A_X}^{-1}\|\,\|W\| \ , \forall \ \ W \in \mathbb{H}^{m\times m}\,,$

$\|\mathbf{Q}W\| \leq \|\mathbf{Q}\|\,\|W\| \ \ , \forall \ \ W \in \mathbb{H}^{m\times m},$

and

$\|\mathbf{P}_Z Y\| \leq \|\mathbf{P}_Z\|\,\|Y\| \ \ , \forall \ \ Y \in \mathbb{C}^{m\times m}\,.$

It is possible then to define the following constants:

$l \ = \ \|\mathbf{L}_{A_X}^{-1}\|^{-1}\,, \qquad p_Z \ = \ \|\mathbf{P}_Z\|\,, \qquad q \ = \ \|\mathbf{Q}\|\,,$

$\varepsilon \ = \ \frac{1}{l}\,\|\Delta Q\| + p_X\|\Delta A\| + q\|\Delta G\| + p_{XS}\|\Delta S\| \quad \varepsilon_1 \ = \ \frac{1}{l}\,\|\Delta S\|_2^2\|X\|,$

$$\delta \ = \ \|\Delta A\| + \|\Delta G\|_2\|X\|\,, \quad \hat{g} \ = \ \|G\|_2 + \|\Delta G\|_2\,, \quad \hat{s} \ = \ \|S\|_2 + \|\Delta S\|_2\,. \tag{9}$$

Further on, we conclude that

$$\hat{s}^2 + 2\delta + 2\sqrt{l\hat{g}(\varepsilon+\varepsilon_1)} \ < l \quad (G \neq 0) \ \Leftrightarrow \ 2\sqrt{l\hat{g}(\varepsilon+\varepsilon_1)} \ < l - \hat{s}^2 - 2\delta$$

$$\Leftrightarrow \ \varepsilon + \varepsilon_1 \ < \ \frac{(l-2\delta-\hat{s}^2)^2}{4l\hat{g}}\,. \tag{10}$$

Thus

$$2\delta + \hat{s}^2 \ < l\,. \tag{11}$$

Theorem 2.1. *Assume that X is the stabilizing solution to (1) and that the condition*

$$\hat{s}^2 + 2\delta + 2\sqrt{l\hat{g}(\varepsilon+\varepsilon_1)} \ < l, \tag{12}$$

holds. Then the perturbed equation (4) has the stabilizing solution $\tilde{X}$ *with*

$$\|X - \tilde{X}\| \leq \frac{2l(\varepsilon+\varepsilon_1)}{l-2\delta-\hat{s}^2+\sqrt{(l-2\delta-\hat{s}^2)^2-4l\hat{g}(\varepsilon+\varepsilon_1)}} \ = \ v_*\,,$$

with constants defined by (9).

Proof. Assume X is the stabilizing solution to (1). According to remark 2.1 the operator $\mathcal{L}$ is c-stable, all eigenvalues of A_X lie in the open left half-plane and $\rho(\mathbf{L}_{A_X}^{-1}\Sigma) < 1$. Then according to (8) we have

$$\Delta X = -\mathbf{L}_{A_X}^{-1}E_1 - \mathbf{L}_{A_X}^{-1}E_2 + \mathbf{L}_{A_X}^{-1}h_1(\Delta X) + \mathbf{L}_{A_X}^{-1}h_2(\Delta X),$$

to obtain

$$\begin{aligned}\Delta X &= -\mathbf{L}_{A_X}^{-1}\Delta Q - \mathbf{P}_X\Delta A + \mathbf{Q}\Delta G - \mathbf{P}_{XS}\Delta S \\ &\quad -\mathbf{L}_{A_X}^{-1}(\Delta S^* X \Delta S) + \mathbf{L}_{A_X}^{-1}h_1(\Delta X) + \mathbf{L}_{A_X}^{-1}h_2(\Delta X) \equiv \mu(\Delta X).\end{aligned} \tag{13}$$

Then

$$\|\Delta X\| = \|\mu(\Delta X)\| \le \varepsilon + \varepsilon_1 + \frac{1}{l}\|h_1(\Delta X)\| + \frac{1}{l}\|h_2(\Delta X)\|,$$

and

$$\begin{aligned}\frac{1}{l}\|h_1(\Delta X)\| + \frac{1}{l}\|h_2(\Delta X)\| &\le \frac{2}{l}(\|\Delta A\| + \|\Delta G\|_2\|X\|)\|\Delta X\| \\ &\quad + \frac{1}{l}(\|S\|_2 + \|\Delta S\|_2)^2\|\Delta X\| + \frac{1}{l}(\|G\|_2 + \|\Delta G\|_2)\|\Delta X\|^2.\end{aligned}$$

Hence we conclude that

$$\|\Delta X\| \le \varepsilon + \varepsilon_1 + \frac{2\delta + \hat{s}^2}{l}\|\Delta X\| + \frac{\hat{g}}{l}\|\Delta X\|^2$$

$$0 \le l(\varepsilon + \varepsilon_1) - (l - 2\delta - \hat{s}^2)\|\Delta X\| + \hat{g}\|\Delta X\|^2.$$

The last inequality lead to the following inequality for ξ.

$$\hat{g}\xi^2 - (l - 2\delta - \hat{s}^2)\xi + l(\varepsilon + \varepsilon_1) \ge 0.$$

Then, the positive scalar ν_*,

$$\nu_* = \frac{2l(\varepsilon + \varepsilon_1)}{l - 2\delta - \hat{s}^2 + \sqrt{(l - 2\delta - \hat{s}^2)^2 - 4l\hat{g}(\varepsilon + \varepsilon_1)}},$$

is the smaller root of the equation

$$\hat{g}\xi^2 - (l - 2\delta - \hat{s}^2)\xi + l(\varepsilon + \varepsilon_1) = 0.$$

If $\|\Delta X\| \le \nu_*$, then

$$\|\mu(\Delta X)\| \le \varepsilon + \varepsilon_1 + \frac{2\delta + \hat{s}^2}{l}\|\Delta X\| + \frac{\hat{g}}{l}\|\Delta X\|^2 \le \nu_*.$$

Now if we define the set

$$\mathcal{S}_{\nu_*} = \{\Delta X \in \mathbb{H}^{m\times m} \ : \ \|\Delta X\| \le \nu_*\},$$

then according to Schauder's fixed point theorem, there exists $\Delta X_* \in \mathcal{S}_{\nu_*}$, such that $\mu(\Delta X_*) = \Delta X_*$. Hence there exists a solution ΔX_* of the equation (13) for which $\|\Delta X_*\| \le \nu_*$.

Furthermore, we will use the following property of ν_*,

$$\nu_* = \frac{2l(\varepsilon+\varepsilon_1)}{l-2\delta-\hat{s}^2+\sqrt{(l-2\delta-\hat{s}^2)^2-4l\hat{g}(\varepsilon+\varepsilon_1)}}$$
$$< \frac{2l(\varepsilon+\varepsilon_1)}{l-2\delta-\hat{s}^2}$$
$$< \frac{2l}{l-2\delta-\hat{s}^2}\frac{(l-2\delta-\hat{s}^2)^2}{4l\hat{g}} = \frac{l-2\delta-\hat{s}^2}{2\hat{g}},$$

which is based on the inequalities (11) and (10).
For $Y = X + \Delta X_*$, we wish to prove that Y is the stabilizing solution of perturbed equation (4). Accordingly

$$(\tilde{A} - \tilde{G}Y)^* Y + Y(\tilde{A} - \tilde{G}Y) = -Y\tilde{G}Y - \tilde{Q} - \tilde{S}^* Y\tilde{S}.$$

First, we will prove that $(\tilde{A} - \tilde{G}Y)$ is stable. For this purpose we write down

$$\tilde{A} - \tilde{G}Y = A_X + E,$$

where $A_X = A - GX$ and $E = \Delta A - \Delta GX - (G + \Delta G)\Delta X_*$.
Thus

$$\|E\| = \|\Delta A - \Delta GX - (G + \Delta G)\Delta X_*\|$$
$$\leq (\|\Delta A\| + \|\Delta G\|_2 \|X\|) + (\|G\|_2 + \|\Delta G\|_2)\|\Delta X_*\|$$
$$\leq \delta + \hat{g}\nu_* < \delta + \hat{g}\frac{l-2\delta-\hat{s}^2}{2\hat{g}} = \frac{l-\hat{s}^2}{2} \leq \frac{l}{2}.$$

Then, we have

$$2\|\mathbf{L}_{A_X}^{-1}\|\,\|E\| < 2\frac{1}{l}\,\frac{l}{2} = 1.$$

So $\tilde{A} - \tilde{G}Y = A_X + E$ is a c-stable matrix because of lemma 2.1.

Furthermore, let us consider

$$\mathbf{L}_{\tilde{A}-\tilde{G}Y}^{-1}\tilde{\Pi}(Y), \quad with \quad \tilde{\Pi}(Y) = \tilde{S}^* Y\tilde{S}.$$

Clearly then

$$\begin{aligned}\mathbf{L}_{\tilde{A}-\tilde{G}Y}(Z) &= (\tilde{A} - \tilde{G}Y)^* Z + Z(\tilde{A} - \tilde{G}Y)\\ &= ((A + \Delta A) - (G + \Delta G)(X + \Delta X_*))^* Z\\ &\quad + Z((A + \Delta A) - (G + \Delta G)(X + \Delta X_*))\\ &= (A - GX)^* Z + Z(A - GX) + [\Delta A - \Delta GX - (G + \Delta G)\Delta X_*]^* Z\\ &\quad + Z[\Delta A - \Delta GX - (G + \Delta G)\Delta X_*]\\ &= \mathbf{L}_{A_X}(Z) + \Delta\mathbf{L}_{A_X}(Z).\end{aligned}$$

Now consider

$$\|\Delta\mathbf{L}_{A_X}\| = \max_{\substack{Z \in \mathbb{H}^{m\times m}\\ \|Z\| \neq 0}} \frac{\|\Delta L_{A_X} Z\|}{\|Z\|}$$

$$\leq 2[\|\Delta A\| + \|\Delta G\|_2 \|X\| + (\|G\|_2 + \|\Delta G\|)\|\Delta X\|_*]$$
$$\leq 2(\delta + \hat{g}\nu_*).$$

Hence

$\|\mathbf{L}_{A_X}^{-1}\| \, \|\Delta \mathbf{L}_{A_X}\| \leq \frac{2}{l}(\delta + \hat{g}\nu_*) < 1.$

Since

$\mathbf{L}_{\tilde{A}-\tilde{G}Y}^{-1} = (\mathcal{I} + \mathbf{L}_{A_X}^{-1}\Delta \mathbf{L}_{A_X})^{-1}\mathbf{L}_{A_X}^{-1}$,

we conclude that

$$\|\mathbf{L}_{\tilde{A}-\tilde{G}Y}^{-1}\,\tilde{\Pi}\| \leq (1 - \|\mathbf{L}_{A_X}^{-1}\|\,\|\Delta \mathbf{L}_{A_X}\|)^{-1}\|\mathbf{L}_{A_X}^{-1}\,\tilde{\Pi}\| \leq \frac{l}{l - 2(\delta + \hat{g}\nu_*)}\,\|\mathbf{L}_{A_X}^{-1}\,\tilde{\Pi}\|\,.$$

As for the $\tilde{\Pi}(Y)$ we have

$$\begin{aligned} \tilde{\Pi}(Y) &= \tilde{S}^* Y \tilde{S} \\ &= S^* Y S + S^* Y \Delta S + \Delta S^* Y S + \Delta S^* Y \Delta S \\ \mathbf{L}_{A_X}^{-1}\,\tilde{\Pi}(Y) &= \mathbf{L}_{A_X}^{-1}\,(S^* Y S) + \mathbf{L}_{A_X}^{-1}(S^* Y \Delta S) + \mathbf{L}_{A_X}^{-1}(\Delta S^* Y S) + \mathbf{L}_{A_X}^{-1}(\Delta S^* Y \Delta S)\,. \end{aligned}$$

This can be used to estimate the norm of operators viz

$$\|\mathbf{L}_{A_X}^{-1}\,\tilde{\Pi}\| = \max_{\substack{Y \in \mathbb{H}^{m\times m} \\ \|Y\| \neq 0}} \frac{\|\mathbf{L}_{A_X}^{-1}\tilde{\Pi}(Y)\|}{\|Y\|} \leq \frac{1}{l}(\|S\|_2 + \|\Delta S\|_2)^2 = \frac{\hat{s}^2}{l}\,.$$

Hence

$\|\mathbf{L}_{\tilde{A}-\tilde{G}Y}^{-1}\,\tilde{\Pi}\| \leq \frac{\hat{s}^2}{l-2(\delta+\hat{g}\nu_*)}\,.$

Moreover, from (12) we have

$\hat{s}^2 + 2(\delta + \hat{g}\nu_*) < l \quad \Leftrightarrow \quad \frac{\hat{s}^2}{l-2(\delta+\hat{g}\nu_*)} < 1\,.$

Thus

$\rho\left(\mathbf{L}_{\tilde{A}-\tilde{G}Y}^{-1}\,\tilde{\Pi}\right) \leq \|\mathbf{L}_{\tilde{A}-\tilde{G}Y}^{-1}\,\tilde{\Pi}\| < 1\,.$

Then

$\mathbf{L}_{\tilde{A}-\tilde{G}Y}(Z) + \tilde{\Pi}(Z) = (\tilde{A} - \tilde{G}Y)^* Z + Z(\tilde{A} - \tilde{G}Y) + \tilde{S}^* Z \tilde{S}$

is c-stable, i.e. $Y = X + \Delta X_*$ is the stabilizing solution of (4). ■

2.3. Numerical experiments

We shall compute the perturbation estimate for equation (1) by using MATLAB, and use the Frobenius matrix norm $\|..\|_F$ as a unitary invariant norm in the pertaining numerical experiments below.

Example 2.1. Consider the matrix equation (1) when

$$A = \frac{2\sqrt{3}}{50}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.25 & 0 & 0 & 0 & 1 \\ 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0.25 & 0 & 0 & 0.25 & 0 \\ 1 & 0.25 & 0 & 0 & 0.25 \end{pmatrix}, \quad G = \begin{pmatrix} 2.2 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 2.2 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 2.2 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 & 2.2 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.05 & 2.2 \end{pmatrix}.$$

The solution is

$$X = diag(1.25, 1, 1, 1, 1) \text{ and } Q = XGX - A^*X - XA - S^*XS.$$

The perturbed equation is

$$\tilde{A}_j^*\tilde{X}_j + \tilde{X}_j\tilde{A}_j - \tilde{X}_j\tilde{G}_j\tilde{X}_j + \tilde{Q}_j + \tilde{S}_j^*\tilde{X}_j\tilde{S}_j = 0, \tag{14}$$

where

$$\epsilon = (0.1)^{2j}, \quad \tilde{A}_j = A + \epsilon(I+E), \quad \tilde{G}_j = G + \epsilon(I+E), \quad \tilde{S}_j = S + \tfrac{1}{5}\epsilon(I+E),$$

$$\tilde{X}_j = X + \epsilon(I-E), \quad \tilde{Q}_j = \tilde{X}_j\tilde{G}_j\tilde{X}_j - \tilde{A}_j^*\tilde{X}_j - \tilde{X}_j\tilde{A}_j - \tilde{S}_j^*\tilde{X}_j\tilde{S}_j,$$

with

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We then substitute $\tilde{A} = \tilde{A}_j, \tilde{S} = \tilde{S}_j$, $\tilde{G} = \tilde{G}_j$, $\tilde{Q} = \tilde{Q}_j$, $\tilde{X} = \tilde{X}_j$ into the above formulas and compute the corresponding perturbation bounds. The results are given in Table 1. Here we have used theorem 2.1 and the bound ν_* for estimating the exact solution X. The condition (12) of theorem 2.1 is satisfied for this example.

Example 2.2. Consider the matrix equation (1) with the same coefficients as in example 1. We compute $\tilde{A}_j,\ \tilde{G}_j,\ \tilde{S}_j$ as in example 2.1 and $\tilde{Q}_j = Q + \epsilon(I + E)$. Equation (14) is then solved via the iterative procedure proposed by Guo [9]. The computed positive definite solution is $\hat{X}$, and the results are given in the same Table 1.

Table 1 : Numerical results.

	$j = 2$	$j = 3$	$j = 4$	$j = 5$
Example 2.1				
true error $\frac{\|\tilde{X}-X\|_F}{\|X\|_F}$	$1.8962e-04$	$1.8962e-06$	$1.8962e-08$	$1.8962e-10$
Theorem 2.1 $\frac{\nu_*}{\|X\|_F}$	$7.6224e-04$	$7.5985e-06$	$7.5983e-08$	$7.5983e-10$
Example 2.2				
true error $\frac{\|\hat{X}-X\|_F}{\|X\|_F}$	$1.2225e-04$	$1.2226e-06$	$1.2227e-08$	$1.2321e-10$
Theorem 2.1 $\frac{\nu_*}{\|X\|_F}$	0.0024	$2.2671e-05$	$2.2655e-07$	$2.2655e-09$

Example 2.3. Define

$$A = \frac{2\sqrt{3}}{50}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.25 & 0 & 0 & 0 & 1 \\ 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0.25 & 0 & 0 & 0.25 & 0 \\ 1 & 0.25 & 0 & 0 & 0.25 \end{pmatrix}, \quad G = \begin{pmatrix} 2.2 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ & 2.2 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ & & 2.2 & \frac{1}{7} & \frac{1}{7} \\ & sym & & 2.2 & \frac{1}{7} \\ & & & & 2.2 \end{pmatrix}$$

with the solution

$X = diag(1.25, 1, 1, 1, 1)$ and $Q = XGX - A^*X - XA - S^*XS$.

The coefficients of the perturbed equation are

$\tilde{A}_j = A + \Delta_j, \; \tilde{G}_j = G + \Delta_j, \; \tilde{S}_j = S + \frac{1}{5}\Delta_j, \; \tilde{Q}_j = Q + \Delta_j$

where

$\Delta_j = 0.1^j (I + E) \; or \; \Delta_j = 0.1^j E, \quad j = 3, 4, \ldots, 12$

Table 2 : Numerical results.

j	true error $\frac{\lVert\hat{X}-X\rVert_F}{\lVert X\rVert_F}$	Theorem 2.1	true error $\frac{\lVert\hat{X}-X\rVert_F}{\lVert X\rVert_F}$	Theorem 2.1
	$\Delta_j = 0.1^j (I+E)$		$\Delta_j = 0.1^j E$	
3	0.0014	0.0058	0.0011	0.0045
4	1.4090e-004	5.5832e-004	1.0651e-004	4.4109e-004
5	1.4092e-005	5.5653e-005	1.0652e-005	4.3994e-005
6	1.4092e-006	5.5635e-006	1.0652e-006	4.3983e-006
7	1.4092e-007	5.5633e-007	1.0652e-007	4.3982e-007
8	1.4092e-008	5.5633e-008	1.0652e-008	4.3982e-008
9	1.4095e-009	5.5633e-009	1.0655e-009	4.3982e-009
10	1.4126e-010	5.5633e-010	1.0681e-010	4.3982e-010
11	1.4437e-011	5.5633e-011	1.0945e-011	4.3982e-011
12	1.7985e-012	5.5633e-012	1.4266e-012	4.3982e-012

with the same E matrix as in example 2.1.

Equation (14) is solved via the iterative procedure proposed by Guo [9]. The computed positive definite solution is $\hat{X}$ and the results are listed in Table 2.

The results of these examples show that the derived perturbation estimate for the stabilizing solution is an approximation to the real stabilizing solution.

3. A Perturbation Bound for the Discrete-Time Case

In this section we will investigate the discrete type Riccati equation (2), and start by transforming it to the form:

$$X - C^* X(I + GX)^{-1} C - Q + S^* XS = 0\,. \qquad (15)$$

Our interest is in how the stabilizing X of (15) is changed when the coefficient matrices (15) are slightly perturbed. The new perturbation equation is :

$$\tilde{X} - \tilde{C}^* \tilde{X}(I + \tilde{G}\tilde{X})^{-1} \tilde{A} - \tilde{Q} + \tilde{S}^* \tilde{X}\tilde{S} = 0\,, \qquad (16)$$

where $\tilde{C} = C + \Delta C,\ \tilde{Q} = Q + \Delta Q,$ with $\Delta Q, \Delta X \in \mathbb{H}^{m\times m}$ and $\Delta C \in \mathbb{C}^{m\times m}$.

3.1. **Preliminaries**

Given a complex square matrix $\Phi,$ the discrete-time Lyapunov operator $\mathcal{D}_\Phi$ is defined by

$$\mathcal{D}_\Phi \ : \mathbb{H}^{m\times m} \to \mathbb{H}^{m\times m}\,, \quad \mathcal{D}_\Phi X = \Phi^* X \Phi\,.$$

Discrete-time Lyapunov operators play an important role in the investigation of discrete-time Riccati equations. This operator is positive and if Φ is nonsingular, then it is also inverse positive. In particular, the identity map $\mathcal{I}$ is positive and is inverse positive. It is well known that if all eigenvalues of Φ lie in the open unit disk then $\mathcal{I} - \mathcal{D}_\Phi$ is inverse positive. The next property of $\mathcal{D}_\Phi$ is to be quite useful later on.

Lemma [17] **3.1**. *Let L_Φ be the linear operator defined by*

$$\mathcal{L}_\Phi := (\mathcal{I} - \mathcal{D}_\Phi)\, W = W - \Phi^* W \Phi, \qquad W \in \mathbb{H}^{m\times m},$$

with a d-stable matrix $\Phi \in C^{m\times m}$ and

$$\ell = \|\mathcal{L}_\Phi^{-1}\|^{-1}, \qquad \phi = \|\Phi\|_2\,. \qquad (17)$$

If $E \in C^{m\times m}$ satisfies

$$\|E\| < \frac{\ell}{\phi + \sqrt{\phi^2 + \ell}}\,,$$

then $\Phi + E$ is a d-stable matrix.

Consider then the linear Stein equation

$$X = \mathcal{D}_\Phi X + \mathcal{D}_S X + Q\,, \qquad (18)$$

where Φ, S and Q are given $m \times m$ matrices, and Q is Hermitian.

As a corollary to Theorem 3.2 of [6] we can formulate the lemma that follows.

Lemma 3.2. *The following statements are equivalent.*

(i) All eigenvalues of Φ lie in the open unit disk and $\rho((I - D_\Phi)^{-1} D_S) < 1$;

(ii) If $Q > 0$ then (18) has a unique solution $X > 0$;

(iii) $D_\Phi + D_S$ *is d-stable.*

Lemma [6] **3.3**. *If* $D_\Phi + D_S$ *is d-stable and* $Q \geq 0$, *then (18) has a unique solution* $X \geq 0$.

We begin with the Fréchet derivative $\mathcal{P}'(X)$ of the matrix function $\mathcal{P}(X)$.

Lemma [7] **3.4**. *For any* X *with* $det[R + B^*XB] \neq 0$ *the Fréchet derivative* $P'_X : H^{m\times m} \to H^{m\times m}$ *is given by* $P'_X = D_{C+BF_X} + D_S$ *with* $F_X = -(R + B^*XB)^{-1}(C^*XB)^*$.

A matrix $X \in \mathbb{H}^{m\times m}$ is called stabilizing for $\mathcal{P}$ if $\sigma(\mathcal{P}'_X)$ lies in open unit disk. In addition if a stabilizing matrix X is a solution to $\mathcal{P}(X) = X$, then X is a stabilizing solution.

Remark 3.1. Following Lemmata 3.2 and 3.4 we can conclude that if X is a stabilizing solution to $\mathcal{P}(X) = X$ then $\mathcal{P}'_X$ is d-stable and thus all eigenvalues of $C + BF_X$ lie in the open unit disk, and $\rho((\mathcal{I} - \mathcal{D}_{C+BF_X})^{-1}\mathcal{D}_S) < 1$. Moreover, the matrix $C + BF_X$ is d-stable relative to $\mathcal{D}_S$. In addition, using the Woodbury matrix identity [13, 17], we have the closed loop system matrix

$$C + BF_X = [I - B(R + B^*XB)^{-1}B^*X]C$$

$$C + BF_X = (I + BR^{-1}B^*X)^{-1}C = (I + GX)^{-1}C.$$

Thus for the stabilizing solution X, this closed loop system matrix $(I + GX)^{-1}C$ is d-stable.

3.2. **A perturbation bound**

According to Sun's paper [17], $\Delta X = \tilde{X} - X$ satisfies the equation

$$\Delta X - \Phi^*\Delta X\Phi = E_1 + E_2 + h_1(\Delta X) + h_2(\Delta X), \tag{19}$$

where

$$\begin{aligned}
F &= (I + GX)^{-1}, \quad \Phi = FC, \\
\Psi &= XF, \quad \Theta = F(I + \Delta G\Psi)^{-1}, \quad K = \Psi C, \\
E_1 &= \Delta Q + K^*\Delta C + \Delta C^*K - K^*\Delta GK + S^*X\Delta S + \Delta S^*XS \\
E_2 &= (\Delta C - \Delta GK)^*\,\Psi\,(I + \Delta GK)^{-1}\,(\Delta C - \Delta GK) - \Delta S^*X\Delta S \\
h_1(\Delta X) &= \Delta\Phi^*\Delta X\Phi + \Phi^*\Delta X\Delta\Phi + \Delta\Phi^*\Delta X\Delta\Phi + (S + \Delta S)^*\Delta X(S + \Delta S) \\
h_2(\Delta X) &= -(C + \Delta C)^*\Theta^*\Delta X\Theta(G + \Delta G)\Delta X\Theta(I + (G + \Delta G)\Delta X\Theta)^{-1}(C + \Delta C).
\end{aligned}$$

Using the following operators

$\mathbf{P}_Z : \mathbb{C}^{m\times m} \to \mathbb{H}^{m\times m}$ by

$$\mathbf{P}_Z N := \mathcal{L}_\Phi^{-1}(Z^*N + N^*Z), \quad Z, N \in \mathbb{C}^{m\times m},$$

$\mathbf{Q} : \mathbb{H}^{m\times m} \to \mathbb{H}^{m\times m}$ by

$$\mathbf{Q}H := \mathcal{L}_\Phi^{-1}(K^*HK), \quad K = \Psi C, \quad H \in \mathbb{H}^{m\times m}.$$

allows for defining the following constants:

$$\left.\begin{array}{l}
\ell = \|\mathcal{L}_\Phi^{-1}\|^{-1}, \quad p_Z = \|\mathbf{P}_Z\|, \quad q = \|\mathbf{Q}\|, \quad \phi = \|\Phi\|_2, \quad \psi = \|\Psi\|_2, \\
\alpha = \|C\|_2, \quad \kappa = \|K\|_2, \quad f = \|F\|_2, \quad g = \|G\|_2, \\
\tilde{s} = \|S\|_2 + \|\Delta S\|_2, \quad x = \|X\|_2, \\
\delta = \dfrac{\|\Delta C\| + \kappa\|\Delta G\|}{1 - \psi\|\Delta G\|_2}, \\
\phi = \|\Phi\|_2, \quad \eta = f\delta(2\phi + f\delta) + \tilde{s}^2, \\
\varepsilon_1 = \frac{1}{\ell}\|\Delta Q\| + p_K\|\Delta C\| + q\|\Delta Q\| + p_{XS}\|\Delta S\|, \\
\varepsilon_2 = \psi\delta(\|\Delta C\|_2 + \kappa\|\Delta G\|_2) + x\|\Delta S\|_2\|\Delta S\|, \\
\varepsilon = \varepsilon_1 + \frac{1}{\ell}\varepsilon_2, \\
\hat{\alpha} = \dfrac{f(\alpha + \|\Delta C\|_2)}{1 - \psi\|\Delta G\|_2}, \quad \hat{g} = \dfrac{f(g + \|\Delta G\|_2)}{1 - \psi\|\Delta G\|_2}.
\end{array}\right\} \tag{20}$$

Theorem 3.1. *Assume that X is the stabilizing solution to (15) and consider the linear operators L_Φ, P_Z and Q described above and the constants given by (20). Let $\tilde{C} = C + \Delta C$, $\tilde{G} = G + \Delta G \geq 0$, $\tilde{Q} = Q + \Delta Q \geq 0$ be the coefficient matrices of the perturbed equation (16). Moreover, if*

$$1 - \psi\|\Delta G\|_2 > 0, \qquad \ell - \eta > 0,$$

$$\varepsilon \leq \frac{(\ell-\eta)^2}{\ell\hat{g}\left(\ell - \eta + 2\hat{\alpha}^2 + \sqrt{(\ell - \eta + 2\hat{\alpha}^2)^2 - (\ell-\eta)^2}\right)}, \tag{21}$$

$$\frac{f\delta + \phi\hat{g}\xi_*}{1 - \hat{g}\xi_*} < \frac{\ell - \tilde{s}^2}{\phi + \sqrt{\phi^2 + \ell - \tilde{s}^2}}, \tag{22}$$

where

$$\xi_* = \frac{2\ell\varepsilon}{\ell - \eta + \ell\hat{g}\varepsilon + \sqrt{(\ell - \eta + \ell\hat{g}\varepsilon)^2 - 4\ell\hat{g}(\ell - \eta + \hat{\alpha}^2)\varepsilon}},$$

then the perturbed equation (16) has the unique stabilizing positive definite solution $\tilde{X}$ with

$$\|X - \tilde{X}\| \leq \xi_*.$$

Proof. Assume X is the stabilizing solution to (15).

According to remark 3.1 the operator $\mathcal{D}_\Phi + \mathcal{D}_S$ is d-stable and all eigenvalues of $\Phi = (I + GX)^{-1}C$ lie in the open unit disk and $\rho(\mathcal{I} - \mathcal{D}_\Phi^{-1}\mathcal{D}_S) < 1$.
We rewrite equation (19) as follows

$$\mathcal{L}_{\Phi}\Delta X = E_1 + E_2 + h_1(\Delta X) + h_2(\Delta X)\,. \tag{23}$$

Thus

$$\xi \Delta X = \mathcal{L}_{\Phi}^{-1} E_1 + \mathcal{L}_{\Phi}^{-1} E_2 + \mathcal{L}_{\Phi}^{-1} h_1(\Delta X) + \mathcal{L}_{\Phi}^{-1} h_2(\Delta X) \equiv \mu(\Delta X),$$

where $\mu(\Delta X)$ is a continuous mapping from $\mathbb{H}^{m\times m}$ to $\mathbb{H}^{m\times m}$.

Define then the set

$$\mathcal{T}_{\xi_*} = \{\Delta X \in \mathbb{H}^{m\times m} \ : \ \|\Delta X\| \le \xi_*\}.$$

Now we will prove that for each $\Delta X \in \mathcal{T}_{\xi_*}$, the map $\mu(\Delta X) \in \mathcal{T}_{\xi_*}$. Thus, under an assumption that $1 - \psi\|\Delta G\|_2 > 0$, the following inequalities hold [17].

$$\|E_2\| \le \psi\delta(\|\Delta A\|_2 + \kappa\|\Delta G\|_2) + x\|\Delta S\|_2\|\Delta S\| = \varepsilon_2\,.$$

$$\|h_1(\Delta X)\| \le \left[f\,\delta(2\phi + f\delta) + \tilde{s}^2\right]\|\Delta X\| = \eta\,\|\Delta X\|\,.$$

The inequality $\ell - \eta > 0$ means that the right hand of (21) is positive. Thus, according to (21) we conclude that ξ_* exists and $\xi_* > 0$.

Moreover,

$$1 - \hat{g}\xi_* \ge 1 - \frac{2\ell\hat{g}\varepsilon}{\ell - \eta + \ell\hat{g}\varepsilon} = \frac{\ell - \eta - \ell\hat{g}\varepsilon}{\ell - \eta + \ell\hat{g}\varepsilon}$$

$$\ge \frac{\ell - \eta - \frac{(\ell-\eta)^2}{\ell - \eta + 2\hat{\alpha}^2}}{\ell - \eta + \ell\hat{g}\varepsilon} = \frac{2(\ell-\eta)\hat{\alpha}^2}{(\ell - \eta + \ell\hat{g}\varepsilon)(\ell - \eta + 2\hat{\alpha}^2)} > 0\,.$$

Then for each $\Delta X \in \mathcal{T}_{\xi_*}$, we have

$$1 - \hat{g}\|\Delta X\| \ge 1 - \hat{g}\xi_* > 0\,.$$

Furthermore,

$$\|h_2(\Delta X)\| \le \frac{\hat{\alpha}^2\hat{g}\|\Delta X\|^2}{1 - \hat{g}\|\Delta X\|}. \tag{24}$$

At this point we need to estimate $\|\mu(\Delta X)\|$. Accordingly

$$\begin{aligned} \|\mathcal{L}_{\Phi}{}^{-1} E_1\| &\le \|\mathcal{L}_{\Phi}{}^{-1}\Delta Q + \mathbf{P}_K\Delta A - \mathbf{Q}\Delta G + \mathbf{P}_{XS}\Delta A\|, \\ &\le \tfrac{1}{\ell}\,\|\Delta Q\| + p_K\,\|\Delta A\| + q\,\|\Delta Q\| + p_{XS}\,\|\Delta S\| = \varepsilon_1\,. \\ \|\mathcal{L}_{\Phi}{}^{-1} E_2\| &\le \tfrac{1}{\ell}\,\varepsilon_2\,. \end{aligned}$$

Moreover

$$\|\mathcal{L}_{\Phi}{}^{-1} h_1(\Delta X) + \mathcal{L}_{\Phi}{}^{-1} h_2(\Delta X)\| \le \tfrac{1}{\ell}\left(\eta\|\Delta X\| + \frac{\hat{\alpha}^2\hat{g}\|\Delta X\|^2}{1 - \hat{g}\|\Delta X\|}\right).$$

Thus

$$\|\mu(\Delta X)\| \le \varepsilon + \tfrac{1}{\ell}\left(\eta\|\Delta X\| + \frac{\hat{\alpha}^2\hat{g}\|\Delta X\|^2}{1 - \hat{g}\|\Delta X\|}\right) \le \varepsilon + \tfrac{1}{\ell}\left(\eta\xi_* + \frac{\hat{\alpha}^2\hat{g}\xi_*^2}{1 - \hat{g}\xi_*}\right) = \xi_*,$$

and $\mu(\mathcal{T}_{\xi_*}) \subset \mathcal{T}_{\xi_*}$.

According to Schauder's fixed point theorem, there exists $\Delta X_* \in \mathcal{T}_{\xi_*}$ such that $\mu(\Delta X_*) = \Delta X_*$. Hence there exists a Hermitian solution ΔX_* of the equation (23) for which $\|\Delta X_*\| \le \xi_*$.

Assume then $Y = X + \Delta X_*$ to prove that Y is the stabilizing solution of perturbed equation (16). We let Y to be a Hermitian solution of the equation (16), viz

$$Y - \tilde{A}^* Y(I + \tilde{G}Y)^{-1}\tilde{A} - \tilde{Q} + \tilde{S}^* Y\tilde{S} = 0 .$$

Section 4.3 of [17] indicates that the above equation is equivalent to

$$Y - \mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}}Y - \mathcal{D}_{S+\Delta S}Y = \tilde{Q} + \tilde{A}^*(I + Y\tilde{G})^{-1}Y\tilde{G}Y(I + \tilde{G}Y)^{-1}\tilde{A}, \tag{25}$$

where

$$(I + \tilde{G}Y)^{-1}\tilde{A} = \Phi + \Phi_1,$$

and

$$\Phi_1 = F(I + G\Delta X_* F + \Delta G\Delta X_* F + \Delta G\Psi)^{-1}(\Delta A - G\Delta X_*\Phi - \Delta G\Delta X_*\Phi - \Delta GK) .$$

Hence

$$\|\Phi_1\| \le \frac{f\delta + \phi\hat{g}\xi_*}{1 - \hat{g}\xi_*} .$$

From (22) we have

$$\frac{f\delta + \phi\hat{g}\xi_*}{1 - \hat{g}\xi_*} < \frac{\ell - \tilde{s}^2}{\phi + \sqrt{\phi^2 + \ell - \tilde{s}^2}} < \frac{\ell}{\phi + \sqrt{\phi^2 + \ell}} ,$$

and by lemma 3.1, it follows that $(I + \tilde{G}Y)^{-1}\tilde{A}$ is d-stable.

It remains to prove that

$$\rho\Big(\big(\mathcal{I} - \mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}}\big)^{-1}\mathcal{D}_{S+\Delta S}\Big) < 1 .$$

For this we consider

$$\begin{aligned}\big(\mathcal{I} - \mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}}\big)Y &= Y - (\Phi + \Phi_1)^* Y(\Phi + \Phi_1)\\ &= Y - \Phi^* Y\Phi - \Phi^* Y\Phi_1 - \Phi_1^* Y\Phi - \Phi_1^* Y\Phi_1\\ &= \mathcal{L}_\Phi Y - \Delta\mathcal{L}Y.\end{aligned}$$

$$\|\Delta\mathcal{L}\| \le 2\|\Phi\|_2\|\Phi_1\|_2 + \|\Phi_1\|_2^2 \le \frac{f\delta + \phi\hat{g}\xi_*}{1 - \hat{g}\xi_*}\left(2\phi + \frac{f\delta + \phi\hat{g}\xi_*}{1 - \hat{g}\xi_*}\right) < \ell .$$

Hence

$$\|\mathcal{L}_\Phi^{-1}\|\,\|\Delta\mathcal{L}\| \le \frac{1}{\ell}\frac{f\delta + \phi\hat{g}\xi_*}{1 - \hat{g}\xi_*}\left(2\phi + \frac{f\delta + \phi\hat{g}\xi_*}{1 - \hat{g}\xi_*}\right) < 1.$$

Since

$$\big(\mathcal{I} - \mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}}\big)^{-1} = (\mathcal{L}_\Phi - \Delta\mathcal{L})^{-1} = (\mathcal{I} - \mathcal{L}_\Phi^{-1}\Delta\mathcal{L})^{-1}\mathcal{L}_\Phi^{-1} ,$$

we conclude that

$$\begin{aligned}\left\| \left(\mathcal{I} - \mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}}\right)^{-1}\mathcal{D}_{S+\Delta S} \right\| &\le \left\| (\mathcal{I} - \mathcal{L}_{\Phi}^{-1}\Delta\mathcal{L})^{-1} \right\| \, \|\mathcal{L}_{\Phi}^{-1}\mathcal{D}_{S+\Delta S}\| \\ &\le \frac{1}{1 - \|\mathcal{L}_{\Phi}^{-1}\|\,\|\Delta\mathcal{L}\|}\, \|\mathcal{L}_{\Phi}^{-1}\|\,\|\mathcal{D}_{S+\Delta S}\| \\ &\le \frac{1}{1 - \frac{1}{\ell}\frac{f\delta+\phi\hat{g}\xi_*}{1-\hat{g}\xi_*}\left(2\phi + \frac{f\delta+\phi\hat{g}\xi_*}{1-\hat{g}\xi_*}\right)}\,\frac{1}{\ell}\,\|\mathcal{D}_{S+\Delta S}\| \\ &= \frac{\|\mathcal{D}_{S+\Delta S}\|}{\ell - \frac{f\delta+\phi\hat{g}\xi_*}{1-\hat{g}\xi_*}\left(2\phi + \frac{f\delta+\phi\hat{g}\xi_*}{1-\hat{g}\xi_*}\right)}\end{aligned}$$

Since $\|\mathcal{D}_{S+\Delta S}\| \le \hat{s}^2$ and in consequence of (22), it follows that

$$\left\| \left(\mathcal{I} - \mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}}\right)^{-1}\mathcal{D}_{S+\Delta S} \right\| \le \frac{\hat{s}^2}{\ell - \frac{f\delta+\phi\hat{g}\xi_*}{1-\hat{g}\xi_*}\left(2\phi + \frac{f\delta+\phi\hat{g}\xi_*}{1-\hat{g}\xi_*}\right)} < 1\,.$$

Therefore

$$\rho\left(\left(\mathcal{I} - \mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}}\right)^{-1}\mathcal{D}_{S+\Delta S}\right) < 1\,.$$

Then

$$\mathcal{D}_{(I+\tilde{G}Y)^{-1}\tilde{A}} + \mathcal{D}_{S+\Delta S}$$

is d-stable and $Y = X + \Delta X_*$ is the stabilizing positive semi-definite solution to (25). ■

3.3. **Numerical experiments**

Here we discuss an example for testing the derived perturbation estimate via theorem 3.5.

Example 3.1. Consider the matrix equation (15) with coefficients:

$$Q = VQ_0V, \quad C = VC_0V, \quad G = VG_0V, \quad S = VS_0V,$$

where

$$Q_0 = diag(10^m, 1, 10^{-m}), \quad C_0 = diag(0, 10^{-m}, 1), \quad G_0 = diag(10^{-m}, 10^{-m}, 10^{-m}),$$

$$S_0 = diag(10^{-m}, 2.\,10^{-m}, 10^{-m}), \quad V = I - 2vv^T/3, \quad v = (1,1,1)^T.$$

The unique positive semidefinite solution X to (15) is given by $X = VX_0V$, where $X_0 = diag(x_1, x_2, x_3)$ with

$$x_i = \{c_i^2 + s_i^2 + q_ig_i - 1 + [(c_i^2 + s_i^2 + q_ig_i - 1)^2 + 4q_ig_i(1 - s_i^2)]^{1/2}\}/[2g_i(1 - s_i^2)],$$

and q_i, c_i, g_i and s_i are the corresponding diagonal elements of Q_0, C_0, G_0 and S_0.

The corresponding perturbed equation (16) has the coefficients:

$$\Delta Q = V\Delta Q_0V, \quad \Delta C = V\Delta C_0V, \quad \Delta G = V\Delta G_0V, \quad \Delta S = V\Delta S_0V,$$

where

$$\Delta Q_0 = \begin{pmatrix} 10^m & -5 & 7 \\ -5 & 1 & 3 \\ 7 & 3 & 10^m \end{pmatrix} \times 10^{-j}, \quad \Delta C_0 = \begin{pmatrix} 3 & -4 & 8 \\ -6 & 2 & -9 \\ 2 & 7 & 5 \end{pmatrix} \times 10^{-j},$$

$$\Delta S_0 = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 3 \end{pmatrix} \times 10^{-j} \quad \text{and} \quad \Delta G_0 = \Delta S_0 \times 10^{-m}.$$

Equation (16) is then solved by the fixed point iteration

$$X_0 = \tilde{Q}, \quad \tilde{X}_{k+1} = \tilde{A}^* \tilde{X}_k (I + \tilde{G}\tilde{X}_k)^{-1} \tilde{A} + \tilde{Q} + \tilde{S}^* \tilde{X}_k \tilde{S}.$$

The computed positive definite solution $\hat{X}$ is with

$$\|\tilde{A}^* \hat{X}(I + \tilde{G}\hat{X})^{-1} \tilde{A} + \tilde{Q} + \tilde{S}^* \hat{X}\tilde{S} - \hat{X}\|_F \leq tol = 10^{-12},$$

and the results are given in Table 3.

Table 3: Numerical results, $m = 2$.

	$j = 10$	$j = 8$	$j = 6$	$j = 4$
true error $\frac{\|\tilde{X}-X\|_F}{\|X\|_F}$	$5.5565e-09$	$5.5568e-07$	$5.5448e-05$	0.0047
Theorem 3.5 $\frac{v_*}{\|X\|_F}$	$9.0389e-09$	$9.0394e-07$	$9.0966e-05$	$*$

4. Conclusion

In this paper we have advanced new perturbation estimates for stabilizing solutions to two kinds of generalized Riccati equations. Numerical experiments show that the new estimates represent a good approximation to the true stabilizing solution.

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