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A New Family of Fourth-Order Iterative Methods for Solving Nonlinear Equations With Multiple Roots

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Abstract. *In this paper, we present new fourth-order methods for finding multiple zeros of nonlinear equations. The method is optimal in the sense of requiring only three evaluations of functions per iteration. It is proved that the method has a convergence of order four. In this way it is demonstrated that the proposed class of methods supports the Kung-Traub hypothesis (1974) on the upper bound* 2^n *of the order of multipoint methods based on* $n+1$ *function evaluations. Numerical examples suggest that the new method is competitive with other fourth-order methods for multiple roots.*

Key words : Newton's Method, Root-Finding, Nonlinear Equations, Multiple Roots, Order of Convergence.

AMS Subject Classifications : 65H05, 41A25

1. Introduction

Finding the roots of nonlinear equations is very important in numerical analysis and has many applications in engineering and other applied sciences. In recent years, some modifications of Newton's classical method for multiple roots have been proposed. Moreover, in this paper, we consider iterative methods to find a multiple root α of multiplicity m, i.e. $f^{(j)}(\alpha) = 0,\ j = 0,1,2,3,\ldots\ldots,m-1$, and $f^{(m)}(\alpha) \neq 0,$ of a nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval I, and it is sufficiently smooth in a neighborhood of α. In this regard, modifications of the Newton method for multiple roots have been proposed and analyzed in [1, 4, 6-8, 11]. However, there are not many methods known to handle the case of multiple roots. Hence we present a fourth-order method for finding multiple zeros of a nonlinear equation which uses only three evaluations of the function per iteration. In fact, we have obtained an optimal order of convergence which supports the Kung and Traub conjecture [3]. Kung and Traub conjecture states that multipoint iteration methods, without memory based on n evaluations, could achieve an optimal convergence order 2^{n-1}. The new fourth-order method has additionally an equivalent efficiency index to the established fourth-order methods presented in [6, 7]. Furthermore, the new method has a better efficiency index than the three-order methods given in [1, 4, 8], and the fourth-order method recently presented in [11]. In view of this fact, the new method is significantly better when compared with the established methods. An indication that the new method is efficient and robust.

The well-known Newton's method for finding multiple roots is given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{2}$$

which converges quadratically [5]. In this paper, we shall use (2) to construct our new fourth-order method.

2. The Method and Analysis of Convergence

In order to establish the order of convergence of the new method we state the following three essential definitions.

Definition 2.1. Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence p is given by

$$\lim_{n\to\infty} \frac{x_n - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0, \tag{3}$$

where ζ is the asymptotic error constant and $p \in \mathbb{R}^+$.

Definition 2.2. Let λ be the number of function evaluations in the root finding method. The efficiency of the new method is measured by the concept of efficiency index [2, 9] and defined as

$$\mu^{1/\lambda}, \tag{4}$$

where μ is the order of the method.

Definition 2.3. Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence [10] may be approximated by

$$COC \approx \frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}, \tag{5}$$

where $n \in \mathbb{N}$.

The new fourth-order method for finding a multiple root of a nonlinear equation is expressed as

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{6}$$

$$x_{n+1} = y_n - mbc\left(\frac{1+abc}{1+(a-2)bc}\right)\frac{f(x_n)}{f'(x_n)}, \tag{7}$$

where $b = \frac{f^{(m)}(y_n)}{f^{(m)}(x_n)}$, $c = \frac{f^{(m-1)}(x_n)}{f^{(m-1)}(y_n)}$, $a \in \mathbb{R}$ and x_0 is the initial value, and provided that the denominators of (6) and (7) are not equal to zero.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function defined on an open interval I, enclosing a multiple zero of $f(x)$. Then the family of iterative methods defined by scheme (7) has fourth-order convergence.*

Proof. Let α be a multiple root of multiplicity m of a sufficiently smooth function $f(x)$, $e = x - \alpha$ and $\check{e} = y - \alpha$, where y is defined by (6).

Using the Taylor expansion of $f(x)$ and $f(y)$ about α , we have

$$f(x_n) = \left(\frac{f^{(m)}(\alpha)}{m!}\right) e_n^m \left[1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots\cdots\right], \tag{8}$$

$$f'(x_n) = \left(\frac{f^{(m)}(\alpha)}{(m-1)!}\right) e_n^{m-1} \left[1 + \left(\frac{m+1}{m}\right)c_1 e_n + \left(\frac{m+2}{m}\right)c_2 e_n^2 + \cdots\cdots\right], \tag{9}$$

where

$$c_k = \frac{m!\, f^{(m+k)}(\alpha)}{(m+k)!\, f^{(m)}(\alpha)} . \tag{10}$$

Moreover by (5), we have

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)} = e_n - e_n\left[1 - \frac{c_1}{m} e_n + \frac{(m+1)c_1^2 - 2mc_2}{m^2} e_n^2 + \cdots\cdots\right], \tag{11}$$

The expansion of $f(y_n)$ about α is

$$f(y_n) = \left(\frac{f^{(m)}(\alpha)}{m!}\right) y_n^m \left[1 + c_1 y_n + c_2 y_n^2 + c_3 y_n^3 + \cdots\cdots\right]. \tag{12}$$

Simplifying (12) leads to

$$f(y_n) = \left(\frac{f^{(m)}(\alpha)}{m!}\right)\left(\frac{c_1}{m}\right)^m e_n^{2m}\left[1 + \left(\frac{2mc_2 - (m+1)c_1^2}{c_1}\right)e_n + \left(\frac{(m^3+3m^2+3m+3)c_1^4 - 2m(2m^2+3m+2)c_1^2c_2 + 4m^2(m-1)c_2^2 + 6m^2 c_1c_3}{2c_1^3}\right)e_n^2 + \cdots\cdots\right]. \tag{13}$$

Also

$$\frac{f(y_n)}{f(x_n)} = \left(\frac{f^{(m)}(\alpha)}{m!}\right)\left(\frac{c_1}{m}\right)^m e_n^m\left[1 + \left(\frac{2mc_2 - (m+2)c_1^2}{c_1}\right)e_n + \cdots\cdots\right]. \tag{14}$$

Since according to (7) we have

$$e_{n+1} = \check{e}_n - mbc\left(\frac{1+abc}{1+(a-2)bc}\right)\frac{f(x_n)}{f'(x_n)}, \tag{15}$$

then substituting appropriate expressions in (15) and after a simplification we obtain the error equation

$$e_{n+1} = 2^{-1}c_1\, m^{-3}\left[2\, c_1^2(1+2b) - m(m-2)(m-3)\right] e_n^4 + \cdots\cdots \tag{16}$$

This error equation establishes the fourth-order convergence of the new method defined by (7). ■

3. The Established Methods

For the purpose of comparison, we consider in this section three fourth-order methods reported recently in [6,7]. Since these methods are well established, we state their essential expressions that are used in order to calculate an approximate solution to given nonlinear equations. This should reveal the effectiveness of the new fourth-order method for multiple roots.

3.1. **The Wu et al. method**

In [11], Wu et al. developed a, now well-established, fourth-order method for finding multiple roots of nonlinear equations. The essential expressions used in the method are

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{17}$$

$$x_{n+1} = y_n - m\frac{f(y_n)}{f'(y_n)}, \tag{18}$$

provided that the denominator of (17) and (18) is not equal to zero.

3.2. **The Sharma et al. method**

In [6], Sharma et al. developed a fourth-order of convergence method with the essential expressions,

$$y_n = x_n - \left(\frac{2m}{m+2}\right)\frac{f(x_n)}{f'(x_n)}, \tag{19}$$

$$x_{n+1} = x_n - \frac{m}{8}\left[k_1 \frac{f(x_n)}{f'(x_n)} - (m-2)k_2 \left\{ (m-1) - k_2 \frac{f'(x_n)}{f'(y_n)} \right\} \frac{f(x_n)}{f'(y_n)} \right],$$

where $k_1 = m^3 - 4m + 8$, $k_2 = (m+2)\left(\frac{m}{m+2}\right)^m$, $n \in \mathbb{N}$, and x_0 is the initial value, and provided that the denominator of (19) is not equal to zero.

3.2. **The Shengguo et al. method**

This is another fourth-order of convergence method that has been developed in [7]. Its essential expressions are,

$$y_n = x_n - \left(\frac{2m}{m+2}\right)\frac{f(x_n)}{f'(x_n)}, \tag{20}$$

$$x_{n+1} = x_n - \left[\frac{\left(\frac{m}{2}\right)(m-2)\left(\frac{m}{m+2}\right)^{-m} f'(y_n) - \left(\frac{m^2}{2}\right) f'(x_n)}{f'(x_n) - \left(\frac{m}{m+2}\right)^{-m} f'(y_n)} \right] \frac{f(x_n)}{f'(x_n)},$$

with $n \in \mathbb{N}$, and x_0 is the initial value, and provided that the denominator of (20) is not equal to zero.

4. Application of the New Fourth-Order Iterative Method

The present fourth-order method described by (7) is employed to solve nonlinear equations and is compared with the Wu et al., the Shengguo et al. and Sharma et al. methods (18), (19) and (20), respectively. To study the performance of the new fourth-order methods, we use ten particular nonlinear equations. We shall determine the consistency and stability of results by examining the convergence of the new iterative methods. The findings are generalized by illustrating the effectiveness of the new methods for determining the multiple root of a nonlinear equation. Consequently, we give estimates of the approximate solution produced by the fourth-order methods and list the errors obtained by each of the methods. The numerical computations listed in the following tables were performed on an algebraic system called Maple. In fact, the errors displayed in the tables are of absolute value and insignificant approximations by the various methods have been omitted.

The new fourth-order method requires three function evaluations and has an order of convergence four. To determine the efficiency index of the new method, we shall employ the definition 2.2. Hence, the efficiency index of the fourth-order methods given is $\sqrt[3]{4} \approx 1.587$, which is identical to other established methods, addressed in this section . The test functions and their exact root α are displayed in Table 1. The difference between the root α and the approximation x_n for test functions with initial guess x_0 , are displayed in Table 2. In fact, x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. In the calculations, 12 TNFE are used by each method. Finally, the computational order of convergence (COC) is displayed in Table 3.

Table 1 : Test functions and their roots.

Function	m	**Root**	**Initial Guess**
$f_1(x) = (x^3 + x + 1)^7$	$m = 7$	$\alpha = -0.68232\ldots$	$x_0 = -0.9$
$f_2(x) = \left(x\, e^{x^2} - \sin(x)^2 + 3\cos x + 5\right)^4$	$m = 4$	$\alpha = -1.20764\ldots$	$x_0 = -1.2$
$f_3(x) = ([x-1]^{10} - 1)^9$	$m = 9$	$\alpha = 0.$	$x_0 = 0.01$
$f_4(x) = \left(e^x + x - 20\right)^{95}$	$m = 95$	$\alpha = 2.84243\ldots$	$x_0 = 3.$
$f_5(x) = (\cos x + x)^{15}$	$m = 15$	$\alpha = -0.73908\ldots$	$x_0 = -1.$
$f_6(x) = (\sin(x)^2 - x^2 + 1)^{500}$	$m = 500$	$\alpha = 1.40449\ldots$	$x_0 = 1.7$
$f_7(x) = \left(e^{-x^2} - e^{x^2} - x^8 + 10\right)^{30}$	$m = 30$	$\alpha = 1.23941\ldots$	$x_0 = 1.3$
$f_8(x) = \left(6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1\right)^{55}$	$m = 55$	$\alpha = -1.57248\ldots$	$x_0 = -2.$
$f_9(x) = (\tan x - e^x - 1)^{11}$	$m = 11$	$\alpha = 1.37104\ldots$	$x_0 = 1.4$
$f_{10}(x) = \left(ln[x^2 + 3x + 5] - 2x + 7\right)^{31}$	$m = 31$	$\alpha = 5.46901\ldots$	$x_0 = 6.$

Table 2 : Comparison of new iterative methods (7).

$f_i(x)$	$b = 0$	$b = 1$	$b = 2$	(18)	(19)	(20)
$f_1(x)$	0.559e-55	0.325e-42	0.582e-38	0.666e-49	0.303e-44	0.162e-44
$f_2(x)$	0.423e-144	0.416e-117	0.603e-111	0.518e-124	0.367e-124	0.304e-125
$f_3(x)$	0.433e-95	0.169e-79	0.263e-74	0.434e-87	0.103e-81	0.693e-67
$f_4(x)$	0.619e-83	0.279e-65	0.441e-60	0.310e-72	0.324e-67	0.322e-67
$f_5(x)$	0.358e-73	0.448e-69	0.200e-66	0.180e-82	0.926e-63	0.901e-63
$f_6(x)$	0.421e-46	0.364e-38	0.980e-35	0.845e-46	0.227e-35	0.227e-35
$f_7(x)$	0.227e-62	0.134e-47	0.523e-43	0.304e-54	0.711e-49	0.677e-49
$f_8(x)$	0.379e-19	0.176e-12	0.218e-10	0.267e-16	0.375e-12	0.372e-12
$f_9(x)$	0.349e-70	0.325e-45	0.341e-40	0.764e-51	0.857e-50	0.511e-50
$f_{10}(x)$	0.170e-124	0.912e-123	0.253e-121	0.197e-144	0.700e-103	0.699e-103

Table 3 : COC of various iterative methods (7).

$f_i(x)$	$b = 0$	$b = 1$	$b = 2$	(18)	(19)	(20)
$f_1(x)$	4.0000	3.9998	3.9992	4.0000	3.9999	3.9999
$f_2(x)$	4.0001	4.0001	4.0000	4.0000	4.0000	4.0000
$f_3(x)$	3.9999	3.9999	4.0001	5.3736	4.0000	3.9999
$f_4(x)$	3.9999	3.9999	4.0001	4.0000	4.0002	4.0001
$f_5(x)$	3.9999	3.9999	3.9998	4.0001	3.9999	3.9999
$f_6(x)$	3.9999	3.9992	3.9983	3.9998	3.9987	3.9987
$f_7(x)$	4.0000	3.9999	3.9998	4.0000	4.0000	4.0000
$f_8(x)$	3.9798	3.9014	3.8292	3.9649	3.8884	3.8884
$f_9(x)$	4.0000	3.9999	3.9996	4.0000	3.9999	3.9999
$f_{10}(x)$	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000

5. Remarks and Conclusion

In this paper, we have introduced a new family of fourth-order iterative methods for solving nonlinear equations with multiple roots. Convergence analysis proves that the new methods preserve the order of convergence. Simply, by introducing new parameters in (7) we have achieved fourth-order of convergence. The prime motive of presenting these new methods was to establish a different approach to obtain a fourth-order of convergence method. We have examined the effectiveness of the new methods by showing the accuracy of the multiple roots of several nonlinear equations. After an extensive experimentation, it can be concluded that the convergence of the tested multipoint methods of the fourth-order is remarkably fast. Furthermore, in most of the test examples, we have empirically found that the best results of the new methods (7) are obtained when $b = 0$.The main purpose of demonstrating the new methods for different types of nonlinear equations was particularly to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and determination of the efficiency of the new iterative method. We have verified numerically that the new methods converge to the optimal order four. Finally, we may conclude that the new iterative methods may be considered a very good alternative to the classical methods.

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