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On Some Additive Functionals of Fractional Brownian Motion as a Doubly Indexed Process

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Abstract. *This paper is divided in two parts. In the first part, we describe a new representation (employing Fourier analysis) for the fractional derivatives of local time of fractional Brownian motion (fBm for brevity). Using this representation we study some regularities of these additive functionals. In the second part we study the bounded p-variation and the bounded p,q-variation in the sense of Young [26], of local time of fBm and also for its fractional derivatives. In the last case, we actually have extended the results of this new representation to a large family of additive functionals of fractional Brownian motion as a doubly indexed process.*

Key words : Fractional Brownian Motion, Fractional Derivative, Local Time, p-Variation, p,q-Variation, Slowly Varying Function.

AMS Subject Classifications : 60J55, 60G22

1. Introduction

In this paper we employ the standard Fourier analysis approach, used by Berman, for the calculation of the moments of local time to give some regularity properties and the p,q-variations of local time of fractional Brownian motion (fBm) and of its fractional derivatives.

A centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ is called fBm with Hurst parameter $H \in (0,1)$ if it has the covariance function:

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{1}$$

This process was first introduced by Kolmogorov [17] and studied by Mandelbrot and Van Ness [18], where a stochastic integral representation in terms of a standard Brownian motion was established. The fBm has the following self-similarity property: For each constant $c > 0$, both processes $\{c^{-H}B^H_{ct}, t \geq 0\}$ and $\{B^H_t, t \geq 0\}$ have the same distribution. This property is an immediate consequence of the fact that the covariance function (1) is homogeneous of order 2H. This process also has stationary increments, and long-range dependence property. These properties make the fBm a suitable driving noise in various areas including mathematical finance, network traffic analysis, turbulence and image processing.

This paper is partially motivated by the following results of the local time of some Markov process. The p-variation of the Brownian local time was first obtained separately by Bouleau and Yor [10] and Perkins [21]. Actually, for a sequence of partitions $\{D_n\}$ of an interval $[a,b]$, with the mesh $|D_n| \to 0$ when $n \to \infty$,

$$\lim_{n\to\infty} \sum_{D_n} (L_t^{x_{i+1}} - L_t^{x_i})^2 = 4\int_a^b L_t^x\, dx$$

in probability. This result allowed to construct various stochastic integrals of the Brownian local time in the spatial variable (See Rogers and Walsh [22]). Marcus and Rosen [19] also extended the results of [10, 21] to the local time of a symmetric stable process with index $1 < \beta \leq 2$. More precisely if $(\pi_n)_{n\in\mathbb{N}}$ is any sequence of partitions of $[a,b] \subset \mathbb{R}$, such that $|\pi_n|$ converges to 0 as n tends to ∞, then for any $0 \leq t \leq T$,

$$\sum_{x_i\in\pi_n} |L_t^{x_{i+1}} - L_t^{x_i}|^{2/(\beta-1)} \xrightarrow[n\to\infty]{L^r} c(\beta)\int_a^b (L_t^x)^{1/(\beta-1)} dx, \quad \text{for all} \quad r > 0,$$

where L_t^x is a symmetric stable local time and $c(\beta)$ is a constant that depends only on β. On another note, Eisenbaum [11] studied this process as a doubly indexed process. For $[a,b]$ a subinterval of $\mathbb{R}$ and $[s,t]$ a subinterval of $[0,+\infty)$ denote a sequence of grids of $[a,b] \times [s,t]$ by

$$\Delta_k = \{(x_i, s_j), 1 \leq i \leq n, 1 \leq j \leq m\}, \qquad \text{for all} \quad k \in \mathbb{N}.$$

Then the following limit holds.

$$\sum_{(x_i,t_j)\in\Delta_k} |L_{t_{j+1}}^{x_{i+1}} - L_{t_j}^{x_{i+1}} - L_{t_{j+1}}^{x_i} + L_{t_j}^{x_i}|^{\frac{2}{\beta-1}} \xrightarrow[k\to\infty]{L^1} 0,$$

where $|\Delta_k|$ and $\underset{(x_i,\, t_j)}{sup} \frac{|x_{i+1}-x_i|}{(t_{j+1}-t_j)^{1/\beta}}$ both converge to 0 as k tends to ∞.

By the same assumptions as of Eisenbaum [11], recently Ait Ouahra et al [2] studied the p,q-variation of fractional derivatives D (see the definition below) of a symmetric stable local time with index $1 < \beta \leq 2$, more precisely

$$\sum_{(x_i,t_j)\in\Delta_k} |DL_{t_{j+1}}^{\cdot}(x_{i+1}) - DL_{t_j}^{\cdot}(x_{i+1})$$

$$-DL_{t_{j+1}}^{\cdot}(x_i) + DL_{t_j}^{\cdot}(x_i)|^{\frac{2}{\beta-2\gamma-1}} \xrightarrow[k\to\infty]{L^1} 0.$$

Another motivation comes from the desire to connect Eisenbaum's [11] results with those of Feng and Zhao [13]. In fact, Feng and Zhao [13] proved that for a continuous semi-martingale, its local time L_t^x is of finite p-variation in the classical sense in x for any $t \geq 0$, a.s.

$$\sup_D \sum_{i=0}^{n-1} |L_t^{x_{i+1}} - L_t^{x_i}|^p < \infty \quad \text{for any} \quad p > 2,$$

where the supremum is taken over all finite partition

$D(-\infty,+\infty) = \{-\infty < x_0 < x_1 < x_2 < \cdots < x_n < +\infty\}$. This allowed to define the path integral $\int_{-\infty}^{\infty} f(x) d_x L_t^x$ as a Young integral, for any f being of a finite q-variation for a number $q \in [1,2)$. They also have shown that for a continuous bounded semi-martingale, its local time L_t^x has a bounded variation uniformly in x, and bounded $2+\varepsilon$-variation uniformly in t, for all $\varepsilon > 0$, then $\int_{-\infty}^{\infty}\int_0^t f(x) d_{t,x} L_t^x$ is well defined (in the sense of Theorem 3.1 [13]), for any f of bounded p,q-variation in (t,x) where $p \geq 1$, $q \geq 1$ and $2q+1 > 2pq$.

The main objective of this work is to give analogous results for the local time of fBm and of its fractional derivative. It should be noted here that the proofs in Eisenbaum [11] and in Ait Ouahra et al [2] depend on the Markov property for a symmetric stable process, then one asks immediately if it can be generalized to a non-Markovian process like fBm. The answer is affirmative. Indeed, the method applied in this work, which is based on the works of Berman ([4, 5, 6]), invokes a Fourier transform representation of fractional derivative of fBm local time .

The paper is organized as follows. In the next section, we present a very brief over view of local time. In section 3, we give a Fourier transform representation of fractional derivatives of fBm local time. We use this representation to give some regularities of this additive functionals. Section 4, is devoted to the study of certain results of bounded p-variation and bounded p,q-variation of local time of fBm and for its fractional derivatives in the sense of Young [26], which allowed to define pathwise stochastic integrals of the forms $\int_0^t f(x) d_x A_s^x$ and $\int_0^t \int_a^b g(s,x) d_{s,x} A_s^x$, for f is α-Hölder, g is (α_1,α_2)-Hölder, and $A_s^x \in \{L_s^x, D^\gamma L_s^\cdot(x)\}$, where the definition of the operator D^γ is given bellow. Subsequently, we give also in this section the p,q-variation of local time of fBm and for its fractional derivatives with Hurst parameter $\frac{1}{2} \leq H < 1$. Finally, in the last section we extend our results to a large family of additive functionals of fractional Brownian motion, more precisely generalized fractional derivative by introducing slowly varying function.

To lighten notations, in the rest of the paper we shall use

$$\Delta_{ij} L := L_{t_{j+1}}^{x_{i+1}} - L_{t_j}^{x_{i+1}} - L_{t_{j+1}}^{x_i} + L_{t_j}^{x_i},$$

and

$$\Delta_{ij} DL := DL_{t_{j+1}}^{\cdot}(x_{i+1}) - DL_{t_j}^{\cdot}(x_{i+1}) - DL_{t_{j+1}}^{\cdot}(x_i) + DL_{t_j}^{\cdot}(x_i).$$

It should be noted here that the paper contains a number of unspecified free positive constants, denoted by C, which may not be the same in each occurrence.

2. Prerequisites

In this section we recall the definition of local time in general at first and we state some regularities of fBm local time that are used all along the paper.

Let $(X(t), t \in \mathbb{R}_+)$ be a separable random process with Borel sample function, the occupation measure of X is defined as follows:

$$\mu(A,B) = \lambda(\{s \in A, X(s) \in B\}), \quad \forall A \in \mathcal{B}(\mathbb{R}_+) \quad \text{and} \quad \forall B \in \mathcal{B}(\mathbb{R}),$$

where λ is the Lebesgue measure on $\mathbb{R}_+$. If $\mu(A,.)$ is absolutely continuous with respect to the

Lebesgue measure on $\mathbb{R}$, we say that X has local time on A and define the local time $L(A,.)$ as the Radon Nikodym derivative of $\mu(A,.)$.

Clearly, the local time L_t^x satisfies the following occupation density formula

$$\int_B f(X_s)ds = \int_{\mathbb{R}} f(x)L(B,x)dx,$$

for every Borel set $B \subset \mathbb{R}_+$ and for all measurable functions $f : \mathbb{R} \to \mathbb{R}_+$. Consequently $x \to L_t^x \in L^1(\mathbb{R})$.

The well-Known Fourier analysis approach for local time of some Gaussian process X introduced by Berman (see [4, 6]), states that for a fixed sample function at fixed t, the Fourier transform on x of L_t^x is the function :

$$F(u) = \int_{\mathbb{R}} e^{iux} L_t^x dx.$$

Using the occupation density formula and the inverse Fourier transform of this function, we have the following representation for local time:

$$L_t^x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^t e^{iu(X(s)-x)}ds\right)du. \tag{2}$$

Xiao [25] proved that the associate family $\{L_t^x,\ t \geq 0,\ x \in \mathbb{R}\}$ of fBm with the Hurst parameter $H \in (0,1)$, has the following regularities

$$\|L_t^x - L_s^x\|_{2m} \leq C|t-s|^{1-H}, \tag{3}$$

and

$$\|L_t^x - L_s^x - L_t^y + L_s^y\|_{2m} \leq C|t-s|^{1-H(1+\xi)}|x-y|^{\xi}, \tag{4}$$

where $0 < \xi < min\,(1, \frac{1}{2H} - \frac{1}{2})$ and $\|.\|_{2m} = [\mathbb{E}|.|^2]^{\frac{1}{2}}$ stands for the norm in $L^{2m}(\Omega)$.

Remark 2.1. The self-similarity property of the fBm immediately implies the scaling property of the local time process:

$$\left(L_{ct}^{c^H x},\ t \geq 0,\ x \in \mathbb{R},\right) \stackrel{\mathfrak{L}}{=} (c^{1-H}L_t^x,\ t \geq 0,\ x \in \mathbb{R}), \quad \text{for any} \quad c > 0. \tag{5}$$

3. Fractional Derivatives of Local Time of fBm and its Regularities

3.1. Fractional derivative of fBm local time

The fractional derivatives and integrals have many uses such as fractional integro-differentiation which has now become a significant topic in mathematical analysis. Fractional derivatives of local time have been discussed for physical purposes in the paper by Ezawa et al [12]. For a complete survey on the fractional integrals and derivatives we refer the reader to Hardy and Littlewood [16] and the book by Samko et al [23].

Let us recall the definition of the fractional derivatives of a real function.

Definition 3.1. Let $0 < \theta < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a function that belongs to $\mathcal{C}^\theta \cap L^1(\mathbb{R})$, where $\mathcal{C}^\theta$ is the space of locally θ –Hölder continuous functions on $\mathbb{R}$. For $0 < \gamma < \theta$, we define $D_\pm^\gamma f$ by:

$$D_{\pm}^{\gamma} f(x) := \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{\infty} \frac{f(x)-f(x \mp y)}{y^{1+\gamma}} dy.$$

The operators D_+^{γ} and D_-^{γ} are called, respectively, right-handed and left-handed Marchaud fractional derivatives of order γ.
We put $D^{\gamma} := D_+^{\gamma} - D_-^{\gamma}$.

In what follows, we give a new representation of fractional derivative of fBm local time, based on Fourier analysis approach which is the main tool in this paper. The regularity (4) of local time of fBm, allows us to define its fractional derivative of order γ:

$$D^{\gamma} L_t^{\cdot}(x) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{\infty} \frac{L_t^{x+a} - L_t^{x-a}}{a^{1+\gamma}} da, \qquad \text{for all } 0 < \gamma < \xi.$$

By using (2), we have then the new representation of fractional derivative of fBm local time:

$$\begin{aligned} D^{\gamma} L_t^{\cdot}(x) &= \frac{\gamma}{2\pi\Gamma(1-\gamma)} \int_0^{\infty} \int_{\mathbb{R}} \int_0^t \frac{e^{iu(X(s)-x+a)} - e^{iu(X(s)-x-a)}}{a^{1+\gamma}} ds du da \\ &= \frac{\gamma}{2\pi\Gamma(1-\gamma)} \int_0^{\infty} \int_{\mathbb{R}} \int_0^t e^{iuX(s)} \left[e^{-iu(x-a)} - e^{-iu(x+a)} \right] \frac{1}{a^{1+\gamma}} ds du da. \end{aligned} \tag{6}$$

Remark 3.1.
(1) The fractional derivatives of the local time is a particular continuous additive functionals of zero energy, in the sense of Fukushima [15].
(2) Like in Fitzsimmons and Getoor [14] for symmetric stable process we have the following scaling property,

$$(D^{\gamma} L_{\lambda t}^{\cdot}(\lambda^H x),\ t \geq 0,\ x \in \mathbb{R}) \stackrel{\mathfrak{L}}{=} (\lambda^{1-(1+\gamma)H} D^{\gamma} L_t^{\cdot}(x),\ t \geq 0,\ x \in \mathbb{R}), \tag{7}$$

for all $\lambda \geq 0$.

3.2. The regularities of the fractional derivative of fBm local time

The new representation (6), allows to give the following regularities of fractional derivative of fBm local time. Throughout this section $\{B^H(t), t \geq 0\}$ is a fBm with the Hurst parameter $H \in (0,1)$.

Theorem 3.1. *Let* $0 < \gamma < \xi$ *and* $D = \{D^{\gamma}, D_{\pm}^{\gamma}\}$. *For all* $m \geq 1$ *there exists a constant* $C > 0$, *such that for every* $0 < t < T$ *and* $(x,y) \in R^2$ *we have*

$$\|DL_t(x) - DL_t(y)\|_{2m} \leq C t^{1-H(1+\xi)} |x-y|^{\xi-\gamma}. \tag{8}$$

Proof. The proof can be given by a slight modifications of the proof in Ait Ouahra et al [2] in the case of a symmetric stable process; and we omit it here. ■

Theorem 3.2. *Under the assumptions of theorem 3.1, there exists a positive constant C such that:*

$$\begin{aligned} &\|DL_t^{\cdot}(x) - DL_t^{\cdot}(y) - DL_s^{\cdot}(x) + DL_s^{\cdot}(y)\|_{2m} \\ &\qquad \leq C|t-s|^{1-H(1+\xi)} |x-y|^{\xi-\gamma}. \end{aligned} \tag{9}$$

Proof. In the following we give a proof of the space-time regularity. To prove this regularity we use our new representation of fractional derivative of fBm local time.

First, consider the following notations:

$$d\bar{u} = \prod_{j=1}^{2m} du_j, \quad d\bar{h} = \prod_{j=1}^{2m} dh_j, \quad d\bar{a} = \prod_{j=1}^{2m} da_j.$$

From (6), for any integer $m \geq 1$, we have:

$$\begin{aligned}
&\| D^\gamma L_t^\cdot(x) - D^\gamma L_t^\cdot(y) - D^\gamma L_s^\cdot(x) + D^\gamma L_s^\cdot(y) \|^{2m} \\
&= C \left\| \int_{[0,+\infty[} \int_{\mathbb{R}} \int_{[s,t]} e^{iuB^H(h)} \left[e^{-iu(x-a)} - e^{-iu(x+a)} - e^{-iu(y-a)} + e^{-iu(y+a)} \right] \frac{1}{a^{1+\gamma}} dhduda \right\|_{2m} \\
&\leq C(I_1 + I_2),
\end{aligned}$$

where

$$I_1 = \left\| \int_{[0,b]} \int_{\mathbb{R}} \int_{[s,t]} e^{iuB^H(h)} \left[e^{-iu(x-a)} - e^{-iu(x+a)} - e^{-iu(y-a)} + e^{-iu(y+a)} \right] \frac{1}{a^{1+\gamma}} dhduda \right\|_{2m},$$

and

$$I_2 = \left\| \int_{[b,+\infty[} \int_{\mathbb{R}} \int_{[s,t]} e^{iuB^H(h)} \left[e^{-iu(x-a)} - e^{-iu(x+a)} - e^{-iu(y-a)} + e^{-iu(y+a)} \right] \frac{1}{a^{1+\gamma}} dhduda \right\|_{2m},$$

for all $b > 0$ and $C = \dfrac{\gamma}{2\pi\Gamma(1-\gamma)}$.

First let us invoke I_1^{2m}:

$$\begin{aligned}
I_1^{2m} &= \left\| \int_{[0,b]} \int_{\mathbb{R}} \int_{[s,t]} e^{iuB^H(h)} \left[e^{-iu(x-a)} - e^{-iu(x+a)} - e^{-iu(y-a)} + e^{-iu(y+a)} \right] \frac{1}{a^{1+\gamma}} dhduda \right\|_{2m} \\
&= \int_{[0,b]^{2m}} \int_{\mathbb{R}^{2m}} \int_{[s,t]^{2m}} \prod_{j=1}^{2m} \left[e^{-iu_j(x-a_j)} - e^{-iu_j(x+a_j)} - e^{-iu_j(y-a_j)} + e^{-iu_j(y+a_j)} \right] \\
&\qquad \bullet \mathbb{E}\left(e^{i\sum_{j=1}^{2m} u_j B^H(h_j)} \right) \prod_{j=1}^{2m} \frac{1}{a_j^{1+\gamma}} d\bar{h} d\bar{u} d\bar{a}.
\end{aligned}$$

Using the elementary inequality $|\ 1 - e^{i\theta}\ | \leq 2^{1-\xi}\ |\ \theta\ |^{\xi}$ for all $0 < \xi < 1$and any $\theta \in \mathbb{R}$, we have

$$\prod_{j=1}^{2m} \left[e^{-iu_j(x-a_j)} - e^{-iu_j(x+a_j)} - e^{-iu_j(y-a_j)} + e^{-iu_j(y+a_j)} \right] \leq C \prod_{j=1}^{2m} |u_j|^{\xi} |a_j|^{\xi}.$$

Hence,

$$
\begin{aligned}
I_1^{2m} &\leq C\int_{[0,b]^{2m}}\int_{\mathbb{R}^{2m}}\int_{[s,t]^{2m}}\prod_{j=1}^{2m}|u_j|^{\xi}\prod_{j=1}^{2m}\frac{|a_j|^{\xi}}{a_j^{1+\gamma}}\mathbb{E}\left(e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right)d\bar{h}d\bar{u}d\bar{a}\\
&\leq C\int_{[0,b]^{2m}}\prod_{j=1}^{2m}a_j^{\xi-1-\gamma}d\bar{a}\int_{\mathbb{R}^{2m}}\int_{[s,t]^{2m}}\prod_{j=1}^{2m}|u_j|^{\xi}\mathbb{E}\left(e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right)d\bar{h}d\bar{u}\\
&\leq Cb^{2m(\xi-\gamma)}\int_{\mathbb{R}^{2m}}\int_{[s,t]^{2m}}\prod_{j=1}^{2m}|u_j|^{\xi}\mathbb{E}\left(e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right)d\bar{h}d\bar{u}, \quad \text{because } \gamma<\xi,\\
&\leq Cb^{2m(\xi-\gamma)}|t-s|^{2m(1-H(1+\xi))},
\end{aligned}
$$

where the last inequality follows from the proof of Lemma 3.3 in Boufoussi et al [8]. Now, we are going to invoke I_2^{2m}.

$$
\begin{aligned}
I_2^{2m} &== \left\|\int_{[b,\infty)}\int_{\mathbb{R}}\int_{[s,t]}e^{iuB^H(h)}\left[e^{-iu(x-a)}-e^{-iu(x+a)}-e^{-iu(y-a)}+e^{-iu(y+a)}\right]\frac{1}{a^{1+\gamma}}dhduda\right\|_{2m}\\
&= \int_{[b,\infty[^{2m}}\int_{\mathbb{R}^{2m}}\int_{[s,t]^{2m}}\prod_{j=1}^{2m}e^{-iu_j(x-a_j)}-e^{-iu_j(x+a_j)}-e^{-iu_j(y-a_j)}+e^{-iu_j(y+a_j)}\\
&\qquad\qquad\bullet\,\mathbb{E}\left(e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right)\prod_{j=1}^{2m}\frac{1}{a_j^{1+\gamma}}d\bar{h}d\bar{u}d\bar{a}.
\end{aligned}
$$

Moreover, applying again the elementary inequality $|1-e^{i\theta}|\leq 2^{1-\xi}|\theta|^{\xi}$ for all $0<\xi<1$ and any $\theta\in\mathbb{R}$, we have:

$$
\prod_{j=1}^{2m}e^{-iu_j(x-a_j)}-e^{-iu_j(x+a_j)}-e^{-iu_j(y-a_j)}+e^{-iu_j(y+a_j)}\leq C\,|x-y|^{2m\xi}\prod_{j=1}^{2m}|u_j|^{\xi}.
$$

Then, we obtain

$$
\begin{aligned}
I_2^{2m} &\leq C\int_{[b,+\infty[^{2m}}\int_{\mathbb{R}^{2m}}\int_{[s,t]^{2m}}|x-y|^{2m\xi}\prod_{j=1}^{2m}|u_j|^{\xi}\prod_{j=1}^{2m}\frac{1}{a_j^{1+\gamma}}\mathbb{E}\left(e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right)d\bar{h}d\bar{u}d\bar{a}\\
&\leq C|x-y|^{2m\xi}\int_{[b,+\infty[^{2m}}\prod_{j=1}^{2m}a_j^{-1-\gamma}d\bar{a}\int_{\mathbb{R}^n}\int_{[s,t]^{2m}}\prod_{j=1}^{2m}|u_j|^{\xi}\mathbb{E}\left(e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right)d\bar{h}d\bar{u}\\
&\leq C|x-y|^{2m\xi}b^{-2m\gamma}|t-s|^{2m(1-H(1+\xi))},
\end{aligned}
$$

where we have used again in the last inequality the arguments used in the proof of Lemma 3.3 in Boufoussi et al [8]. Furthermore, by choosing $b=|x-y|$, we deduce that

$$\| D^{\gamma}L_t^{\cdot}(x) - D^{\gamma}L_t^{\cdot}(y) - D^{\gamma}L_s^{\cdot}(x) + D^{\gamma}L_s^{\cdot}(y) \|_{2m} \leq C\,|t-s|^{1-H(1+\xi)}|x-y|^{\xi-\gamma},$$

which completes the proof of this theorem. ■

Incidentally, by applying Kolmogorov's criterion we have the following mixed Hölder regularity.

Lemma 3.1. *Let $T > 0$, then the following condition is true almost surely. For all $0 < \beta_1 < 1 - H(1+\xi)$ and $0 < \beta_2 < \xi - \gamma$, there is a constant $C > 0$ for any $(t,s) \in [0,T]^2$ and $(x,y) \in R^2$ such that:*

$$| DL_t^{\cdot}(x) - DL_t^{\cdot}(y) - DL_s^{\cdot}(x) + DL_s^{\cdot}(y) | \leq C|t-s|^{\beta_1}|x-y|^{\beta_2}.$$

Remark 3.2.
a) Similar regularity of (8) is given in Marcus and Rosen [19] for the local time of a symmetric stable process with index $1 < \beta \leq 2$. Using this regularity and the Markov property, Ait Ouahra and Eddahbi [1] have demonstrated its mixed Hölder regularity. Notice that a similar result for mixed regularity of local time can be found in Boufoussi and Kamont [9] for the Brownian motion case, and in Xiao [25] for the local time of fractional Brownian motion; and for the local time of the multifractional Brownian motion in Boufoussi et al[8].
b) Recently the mixed Hölder regularity of the fractional derivative of local time of a symmetric stable process is proved in Ait Ouahra et al [2].

4. p,q-variation

In this section, we deal with some results about p,q-variations of local time of fBm and of its fractional derivative with a Hurst parameter $1/2 \leq H < 1$.

4.1. p,q-**Variation of local time**

First, let us recall that a real function $(x,y) \to F(x,y)$, defined on the rectangle $E = [a,b] \times [c,d]$ is said to be of bounded p-variation in x uniformly in y, if

$$\sup_y \sup_{\pi_n} \sum_{i=1}^{n} |F(x_i,y) - F(x_{i+1},y)|^p < \infty,$$

where $\pi_n = \{a = x_0 < x_1 <\ldots< x_n = b\}$ is an arbitrary partition of $[a,b]$, and furthermore, it's of bounded p,q-variation in (x,y), if

$$\sup_{\pi_n} \sup_{\pi'_m} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |\Delta_{i,j}F(x_i,y_j)|^p \right)^q < \infty, \tag{10}$$

for all partitions $E_{n,m} := \pi_n \times \pi_m := \{a = x_0 < x_1 <\ldots< x_n = b;\ c = y_0 < y_1 <\ldots< y_m = d\}$ of $[a,b] \times [c,d]$.

The local time L possesses the following properties.

Theorem 4.1.
i) For all $\varepsilon > 0$, $x \to L_t^x$ is of bounded $\varepsilon + \frac{1}{\min\left(1, \frac{1}{2H} - \frac{1}{2}\right)}$-variation uniformly in t.
ii) L_t^x is of bounded variation in t uniformly in x.
iii) For any $p,q > 0$, such that $\frac{1}{pq} < \xi$ and $\frac{1}{p} < 1 - H(1+\xi)$, L is of bounded p,q-variation in (t,x).

Proof. i) Is the immediate consequence of the fact that every β-Hölder function is of finite $\frac{1}{\beta}$-variation on every compact interval and the following regularity: almost surely, there exist a constant $C > 0$ such that

$$|L_t^x - L_t^y| < C|x-y|^{\xi}, \quad \text{for any} \quad 0 < \xi < \min(1, \tfrac{1}{2H} - \tfrac{1}{2}).$$

ii) Is trivial because $t \to L_t^x$ is an increasing process.
iii) By applying Kolomogrov's criterion to (4), it follows that for every $0 < \beta_1 < 1 - H(1+\xi)$ and $0 < \beta_2 < \xi$ there exist a constant $C > 0$ such that

$$|L_t^x - L_s^x - L_t^y + L_s^y| \leq C\,|t-s|^{\beta_1}|x-y|^{\beta_2},$$

whenever $s,t \in [0,1]$.
Hence (10) in the case of local time of fBm is finite, only when $\frac{1}{pq} < \xi$ and $\frac{1}{p} < 1 - H(1+\xi)$.
This completes the proof of this theorem. ■

Remark 4.1. The integral $\int_a^b F(x) d_x L_t^x$ is well defined as a pathwise Riemann-Stieltjes integral for any function F of q-variation with $1 \leq q < 1 + \frac{\min\left(1, \frac{1}{2H} - \frac{1}{2}\right)}{1 + (\varepsilon - 1)\min\left(1, \frac{1}{2H} - \frac{1}{2}\right)}$, for every $\varepsilon > 0$.

In the rest of the paper, let us consider that
$E_{n,m} := \pi_n \times \pi_m := \{0 = x_0 < x_1 < \ldots < x_n = t;\ c = y_0 < y_1 < \ldots < y_m = d\}$ of $[0,t] \times [c,d]$.
And the following theorem gives the class of functions for which the two parameter p,q-variation path integration w.r.t. local time will be defined.

Theorem 4.2. *Let $E = [0,t] \times [c,d]$ and $0 < \xi < \min(1, \frac{1}{2H} - \frac{1}{2})$. Then almost surely the two parameter integral of local time of fBm,*

$$\int_0^t \int_c^d f(s,x) dL_s^x := \lim_{|E_{n,m}| \to 0} \sum_{i=0}^{n} \sum_{j=0}^{m} f(t_i, x_j) \Delta_{i,j} L,$$

is well defined, for any function $f \in C_E^{\alpha_1,\alpha_2}$ (the space of mixed Hölder function on E), with $1 \geq \alpha_1 > H(1+\xi)$ and $1 \geq \alpha_2 > 1 - \xi$. Moreover, there exist a constant $C(\alpha_1,\alpha_2,\beta_1,\beta_2) > 0$ such that the following inequality

$$\left| \int_0^t \int_c^d f(s,x) dL_s^x \right| \leq C(\alpha_1,\alpha_2,\beta_1,\beta_2) t^{\beta_1} (d-c)^{\beta_2},$$

with $0 < \beta_1 < 1 - H(1+\xi)$ and $0 < \beta_2 < \xi$, is satisfied.

Proof. Applying the two parameter Kolmogorov's criterion of (4), we obtain

$|L_t^x - L_s^x - L_t^y + L_s^y| \leq C|t-s|^{\beta_1}|x-y|^{\beta_2},$

with $0 < \beta_1 < 1 - H(1+\xi)$ and $0 < \beta_2 < \xi$.
Using now Theorem 3.2 in C. Tudor and M. Tudor [24], we arrive at the required result. ■

Remark 4.2.
1) Since the local time has a compact support, for any function $f \in \mathcal{C}_E^{\alpha_1,\alpha_2}$, with $1 \geq \alpha_1 > H(1+\xi)$ and $1 \geq \alpha_2 > 1-\xi$, we can define the integral $I_t := \int_0^t \int_{\mathbb{R}} f(s,x) dL_s^x$, precisely

$$I_t = \lim_{|En,m| \to 0} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(t_i, x_j) \Delta_{i,j} L,$$

2) In the case of Brownian motion, the pathwise integral I_t is defined for any function $f \in \mathcal{C}_E^{\alpha_1,\alpha_2}$ such that $1 \geq \alpha_1 > \frac{3}{4}$ and $1 \geq \alpha_2 > \frac{1}{2}$. In the symmetric stable process of index $1 < \alpha \leq 2$ this integral is defined for any function $f \in \mathcal{C}_E^{\alpha_1,\alpha_2}$ with $1 \geq \alpha_1 > \frac{\alpha+1}{2\alpha}$ and $1 \geq \alpha_2 > \frac{3-\alpha}{2}$.

Now, in the following theorem we give the p,q-variation of local time of fBm with the Hurst parameter $\frac{1}{2} \leq H < 1$, similar to the Eisenbaum's [11] result in the case of a symmetric stable process.

Theorem 4.3. *Let* $1/2 \leq H < 1$, $(\Delta_k)_{k\in\mathbb{N}}$ *be a sequence of grids of* $[a,b] \times [s,t]$, *where for each* k, $\Delta_k = \{(x_i,s_j), 1 \leq i \leq n, 1 \leq j \leq m\}$. *And suppose that* $|\Delta_k|$ *and* $\sup_{(x_i,s_j)\in\Delta_k} \frac{|x_{i+1}-x_i|}{(t_{j+1}-t_j)^H}$ *both converge to* 0 *as k tends to* ∞*,then*

$$\sum_{(i,j)} |\Delta_{ij} L|^{\frac{1}{\xi}} \xrightarrow[k\to\infty]{L^1} 0,$$

where $0 < \xi < \frac{1}{2H} - \frac{1}{2}$.

To prove this result we need the following lemma.

Lemma 4.1. *Let* $H \in [\frac{1}{2}, 1)$. *Then for any* $t \in R$, *we have*

$$\sum_{n\in\mathbb{N}} P[\max_{t\leq s\leq t+1} B_s^H \geq n] < C < +\infty\ ,$$

where the constant C doesn't depend on t.

Proof. We have by Lemma 2.3 in Matsui and Shieh [20], for any $H \in [\frac{1}{2}, 1)$ and $\lambda \geq 0$,

$$\mathbb{P}[\max_{t\leq s\leq t+1} B_s^H \geq \lambda] < \sqrt{\frac{2}{\pi}} \int_\lambda^{+\infty} e^{-\frac{x^2}{2}} dx.$$

Making the change of variable $x = \lambda + u$, we obtain for any $\lambda > 0$

$$\int_\lambda^{+\infty} e^{-\frac{x^2}{2}} dx \leq \frac{\sqrt{2\pi}}{2} e^{-\frac{\lambda^2}{2}}.$$

Finally

$$\sum_{n\in\mathbb{N}} \mathbb{P}[\max_{t\le s\le t+1} B_s^H \ge n] \le \sum_{n\in\mathbb{N}} e^{-\frac{n^2}{2}} < \infty. \qquad \blacksquare$$

4.2. **Proof of theorem 4.3**

For a fixed j, we define

$$C_j^H = \frac{1}{(t_{j+1}-t_j)^H}, \quad d_i = d(i,j) =: \frac{x_i}{(t_{j+1}-t_j)^H},$$

and by applying the scaling property, we have

$$\Delta_{ij}L \stackrel{\mathfrak{L}}{=} C_j^{H-1}\Big[L_{C_jt_{j+1}}^{C_j^H x_{i+1}} - L_{C_jt_{j+1}}^{C_j^H x_i} - L_{C_jt_j}^{C_j^H x_{i+1}} + L_{C_jt_j}^{C_j^H x_i}\Big]$$

$$= C_j^{H-1}\int_{\mathbb{R}}\int_0^1 e^{iuB^H(s+C_jt_j)}\Big[e^{-iuC_j^H x_{i+1}} - e^{-iuC_j^H x_i}\Big]dsdu.$$

Let's denote,

$$l_t^x = \int_{\mathbb{R}}\int_0^t e^{iuB^H(s+C_jt_j)}e^{-iux}dsdu,$$

the local time (modulo a constant $\frac{1}{2\pi}$) of the Gaussian process $X^H(s) = B^H(s+C_jt_j)$ (see (2)). Then

$$\Delta_{i,j}L = C_j^{H-1}\Big[l_1^{C_j^H x_{i+1}} - l_1^{C_j^H x_i}\Big] = C_j^{H-1}\Big[l_1^{d_{i+1}} - l_1^{d_i}\Big]$$

$$= (t_{j+1}-t_j)^{1-H}\,(l_1^{d_{i+1}} - l_1^{d_i}).$$

Therefore we have,

$$\mathbb{E}\Big[\sum_{i,j}|\Delta_{ij}L|^{\frac{1}{\xi}}\Big] = \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}}\,\mathbb{E}\Big[\sum_i |l_1^{d_{i+1}} - l_1^{d_i}|^{\frac{1}{\xi}}\Big]$$

$$= \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}}\,\mathbb{E}\Big[\sum_{d_i\ge 0} |l_1^{d_{i+1}} - l_1^{d_i}|^{\frac{1}{\xi}}\Big]$$

$$+ \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}}\,\mathbb{E}\Big[\sum_{d_i\le 0} |l_1^{d_{i+1}} - l_1^{d_i}|^{\frac{1}{\xi}}\Big]$$

$$=: d_+ + d_-$$

Let's put

$$S_1 = \max_{0\le s\le 1} X_s^H \quad \text{and} \quad I_1 = \min_{0\le s\le 1} X_s^H,$$

and estimate d_+ and d_- separately. Then consider (p,q) as a pair of positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, to apply successively Hölder's inequality and (4), to write

$$
\begin{aligned}
d_+ &= \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}} \mathbb{E}[\sum_{d_i\geq 0} |l_1^{d_{i+1}} - l_1^{d_i}|^{\frac{1}{\xi}} , S_1 \geq d_i] \\
&\leq \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}} \sum_{d_i\geq 0} (\mathbb{E}|l_1^{d_{i+1}} - l_1^{d_i}|^{\frac{p}{\xi}})^{\frac{1}{p}} \mathbb{P}[S_1 \geq d_i]^{\frac{1}{q}} \\
&\leq C(H) \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}} \sum_{d_i\geq 0} (d_{i+1}-d_i) \mathbb{P}[S_1 \geq d_i] \\
&= C(H) \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}} \sum_{d_i\geq 0} (d_{i+1}-d_i) \mathbb{P}[\max_{0\leq s\leq 1}(B^H(s+C_j t_j) \geq d_i]^{\frac{1}{q}} \\
&\leq C(H) \sup_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}-1} \sum_j (t_{j+1}-t_j) \sum_n \sum_{n\leq d_i\leq n+1} (d_{i+1}-d_i) \\
&\quad \cdot \mathbb{P}[\max_{C_j t_j\leq u\leq 1+C_j t_j} B^H(u) \geq d_i]^{\frac{1}{q}} \\
&\leq C(H) \sup_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}-1} \sum_j (t_{j+1}-t_j) \sum_n \mathbb{P}[\max_{C_j t_j\leq u\leq 1+C_j t_j} B^H(u) \geq n]^{\frac{1}{q}}.
\end{aligned}
$$

Since $\frac{1}{2} \leq H < 1$, we have $\frac{1-H}{\xi} - 1 > 0$, then

$$\sup_j \ (t_{j+1}-t_j)^{\frac{1-H}{\xi}-1} \to 0 \quad \text{as} \quad k \to +\infty.$$

Now using lemma 4.1, we deduce that

$$\sum_j (t_{j+1}-t_j) \sum_n \mathbb{P}[\max_{C_j t_j\leq u\leq 1+C_j t_j} B^H(u) \geq n]^{\frac{1}{q}} < C(t-s).$$

Hence

$$d_+ \to 0 \quad \text{as} \quad k \to +\infty.$$

Now, for d_-. We do the same estimations as above by using I_1instead of $S_1$and by symmetry, to arrive at

$$d_- \to 0 \quad \text{as} \quad k \to +\infty.$$

Consequently, we have

$$\mathbb{E}\left[\sum_{i,j} |\Delta_{ij} L|^{\frac{1}{\xi}} \right] \to 0 \quad \text{as} \quad k \to +\infty. \qquad \blacksquare$$

4.3. p,q-**Variation of fractional derivative of local time**

As earlier we give in this subsection some results about integration w.r.t. fractional derivative of local time. The fractional derivative of fBm local time has the following properties.

Theorem 4.4.
i) For any $\varepsilon > 0$, $DL_t^{\cdot}(x)$ *is of bounded* $\varepsilon + \frac{1}{\min\left(1,\frac{1}{2H}-\frac{1}{2}\right)-\gamma}$- *variation in x uniformly in t.*
ii) For any $\varepsilon > 0$, $DL_t^{\cdot}(x)$ *is of bounded* $\varepsilon + \frac{1}{1-H(1+\xi)}$*-variation in t uniformly in x.*
iii)For any $p,q > 0$, *such that* $\frac{1}{pq} < \xi - \gamma$ *and* $\frac{1}{p} < 1 - H(1+\xi)$, *DL is of bounded* p,q*-variations in* (t,x).

Proof . i) By applying Kolomogrov′s criterion to (8), we guarantee almost surely, for every $0 < \delta < \xi - \gamma$, the existence of a constant $C > 0$ such that

$$|DL_t^{\cdot}(x) - DL_t^{\cdot}(y)| \le C\,|x-y|^{\delta}.$$

And by the fact that every β-Hölder function is of finite $1/\beta$-variation on every compact interval, we deduce that $DL_t^{\cdot}(x)$is of p-variation in $x$uniformly in t, for any $p > \frac{1}{\xi-\gamma}$.
ii) Is trivial by Theorem 5.1. of Ait Ouahra and Ouali [3]. Indeed, for every $0 < \alpha < 1 - H(1+\xi)$, we have

$$|D^{\gamma}L_t^{\cdot}(x) - D^{\gamma}L_s^{\cdot}(x)| \le C\,|t-s|^{\alpha}.$$

Consequently, the result follows.
iii) In view of lemma 3.1, we deduce that the fractional derivative is of bounded p,q-variation if $\frac{1}{pq} < \xi - \gamma$and $\frac{1}{p} < 1 - H(1+\xi)$. This completes the proof . ■

Remark 4.3. The integral $\int_a^b F(x)dD^{\gamma}L_t^{\cdot}(x)$is well defined as a pathwise Riemann-Stieltjes integral for any function $F$of q –variation such that $1 \le q < 1 + \frac{\xi-\gamma}{1+(\mathcal{E}-1)(\xi-\gamma)}$, for any $\varepsilon > 0$.

Theorem 4.5. *Let* $E = [0,t] \times [c,d]$ *and* $0 < \xi < \min(1, \frac{1}{2H} - \frac{1}{2})$. *Then almost surely the two parameter integral of fractional derivative of local time of fBm*

$$\int_0^t\int_c^d f(s,x)dD^{\gamma}L_s^{\cdot}(x) := \lim_{|En,m|\to 0}\sum_{i=0}^{n}\sum_{j=0}^{m} f(t_i,x_j)\Delta_{i,j}D^{\gamma}L,$$

is well defined, for any function $f \in C_E^{\alpha_1,\alpha_2}$*(the space of mixed Hölder function on E), with* $1 \ge \alpha_1 > H(1+\xi)$ *and* $1 \ge \alpha_2 > 1 - \xi + \gamma$. *Moreover, there exists a constant* $C(\alpha_1,\alpha_2,\beta_1,\beta_2) > 0$ *such that the following inequality*

$$\left|\int_0^t\int_c^d f(s,x)dD^{\gamma}L_s^{\cdot}(x)\right| \le C(\alpha_1,\alpha_2,\beta_1,\beta_2)t^{\beta_1}(d-c)^{\beta_2},$$

with $0 < \beta_1 < 1 - H(1+\xi)$ *and* $0 < \beta_2 < \xi - \gamma$, *is satisfied.*

Proof. For $0 < \beta_1 < 1 - H(1+\xi)$and $0 < \beta_2 < \xi - \gamma$we have $D^{\gamma}L \in \mathcal{C}^{\beta_1,\beta_2}$. Indeed by lemma 3.1 we have almost surely for all $0 < \beta_1 < 1 - H(1+\xi)$and $0 < \beta_2 < \xi - \gamma$,

$$|D^{\gamma}L_t^{\cdot}(x) - D^{\gamma}L_s^{\cdot}(x) - D^{\gamma}L_t^{\cdot}(y) + D^{\gamma}L_s^{\cdot}(y)| \le C|t-s|^{\beta_1}|x-y|^{\beta_2}.$$

Consequently, using Theorem 2.3 in C.Tudor and M.Tudor [24], we deduce our theorem. ■

Remark 4.4. For any function $f \in \mathcal{C}_E^{\alpha_1,\alpha_2}$, with compact support, where $1 \ge \alpha_1 > H(1+\xi)$and

$1 \geq \alpha_2 > 1 - \xi + \gamma$, we can define the integral $J_t := \int_0^t \int_{\mathbb{R}} f(s,x) dD^\gamma L_s^{\cdot}(x)$, precisely

$$J_t = \lim_{|E_{n,m}| \to 0} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(t_i, x_j) \Delta_{i,j} D^\gamma L.$$

By the same technique as in theorem 4.3, we prove in the next theorem the p,q-variation of the fractional derivative of fBm local time with a Hurst parameter $\frac{1}{2} \leq H < 1$.

Theorem 4.6. *Let* $1/2 \leq H < 1$*and* $(\Delta_k)_{k \in \mathbb{N}}$*be a sequence of grids of* $[a,b] \times [s,t]$, *where for each k,* $\Delta_k = \{(x_i, s_j), 1 \leq i \leq n, 1 \leq j \leq m\}$. *We suppose that* $|\Delta_k|$ *and* $\sup_{(x_i,s_j) \in \Delta_k} \frac{|x_{i+1} - x_i|}{(t_{j+1} - t_j)^H}$ *both converge to* 0 *as k tends to* ∞. *Then for* $D = \{D^\gamma, D_+^\gamma, D_-^\gamma\}$*we have*

$$\sum_{(i,j)} |\Delta_{ij} DL|^{\frac{1}{\xi - \gamma}} \xrightarrow[k \to \infty]{L^1} 0, \tag{11}$$

where $0 < \gamma < \xi$.

Proof. The proof of this theorem is done in several steps, in which we adhere to the same notation as in theorem 4.3.

Step1/$\circ$

Firstly we prove (11)for $D = D_+^\gamma$. By applying the scaling properties (7) and (6) and using the same steps as in the proof of theorem 4.3 we may write

$$\mathbb{E}\left[\sum_{i,j} |\Delta_{ij} D_+^\gamma L|^{\frac{1}{\xi-\gamma}}\right] = \sum_j C_j^{\frac{1-(1+\gamma)H}{\xi-\gamma}} \mathbb{E}\Bigg[\sum_i \Big| \int_0^\infty \int_{\mathbb{R}} \int_0^1 e^{iuX(s+C_j t_j)} \Big(e^{-iu(C_j^H x_{i+1})}$$
$$- e^{-iu(C_j^H x_{i+1} + a)} - e^{-iu(C_j^H x_i)} + e^{-iu(C_j^H x_i + a)}\Big) \frac{1}{a^{1+\gamma}} dsduda \Big|^{\frac{1}{\xi-\gamma}}\Bigg].$$

Next define

$$D^\gamma l_t^{\cdot}(x) = \int_0^\infty \int_{\mathbb{R}} \int_0^t e^{iuB^H(s+C_j t_j)} (e^{-iu(x-a)} - e^{-iu(x+a)}) \frac{1}{a^{1+\gamma}} dsduda.$$

Clearly the fractional derivative of local time of the Gaussian process $X^H(s) = B^H(s + C_j t_j)$ must satisfy

$$\begin{aligned}
\mathbb{E}\left[\sum_{i,j} |\Delta_{ij} D_+^\gamma L|^{\frac{1}{\xi-\gamma}}\right] &= \sum_j (t_{j+1} - t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}} \sum_i \mathbb{E}\left[|D_+^\gamma l_1^{\cdot}(d_{i+1}) - D_+^\gamma l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}}\right] \\
&= \sum_j (t_{j+1} - t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}} \sum_{d_i \geq 0} \mathbb{E}\left[|D_+^\gamma l_1^{\cdot}(d_{i+1}) - D_+^\gamma l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}}\right] \\
&+ \sum_j (t_{j+1} - t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}} \sum_{d_i \leq 0} \mathbb{E}\left[|D_+^\gamma l_1^{\cdot}(d_{i+1}) - D_+^\gamma l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}}\right] \\
&= d'_+ + d'_-.
\end{aligned}$$

We need to represent d'_+ and d'_- separately. However, as in the case of local time, we will deal only with d_+.

$$\mathbb{E}\left[\sum_{d_i\geq 0}|D_+^{\gamma}l_1^{\cdot}(d_{i+1})-D_+^{\gamma}l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}}\right]$$
$$=\mathbb{E}\left[\sum_{d_i\geq 0}|D_+^{\gamma}l_1^{\cdot}(d_{i+1})-D_+^{\gamma}l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}};S_1\geq d_i\right]$$
$$+\mathbb{E}\left[\sum_{d_i\geq 0}|D_+^{\gamma}l_1^{\cdot}(d_{i+1})-D_+^{\gamma}l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}};S_1< d_i\right].$$

Since the local time has a compact support,

$$l_t^x=0\ if\, x\notin\left[\inf_{0\leq s\leq t}X_s^H,\sup_{0\leq s\leq t}X_s^H\right],$$

it follows that

$$\mathbb{E}\left[\sum_{d_i\geq 0}|D_+^{\gamma}l_1^{\cdot}(d_{i+1})-D_+^{\gamma}l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}}\right]$$
$$=\sum_{d_i\geq 0}\mathbb{E}\left[\left|\frac{\gamma}{\Gamma(1-\gamma)}\int_0^{\infty}\frac{l_1^{d_{i+1}+a}-l_1^{d_i+a}-l_1^{d_{i+1}}+l_1^{d_i}}{a^{1+\gamma}}da\right|^{\frac{1}{\xi-\gamma}},S_1\geq d_i\right]$$
$$+\sum_{d_i\geq 0}\mathbb{E}\left[\left|\frac{\gamma}{\Gamma(1-\gamma)}\int_0^{\infty}\frac{l_1^{d_{i+1}+a}-l_1^{d_i+a}-l_1^{d_{i+1}}+l_1^{d_i}}{a^{1+\gamma}}da\right|^{\frac{1}{\xi-\gamma}},S_1,< d_i\right].$$

It is clear that the second term on the right hand side equal to zero. Accordingly,

$$\mathbb{E}\left[\sum_{d_i\geq 0}|D_+^{\gamma}l_1^{\cdot}(d_{i+1})-D_+^{\gamma}l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}}\right]=\mathbb{E}\left[\sum_{d_i\geq 0}|D_+^{\gamma}l_1^{\cdot}(d_{i+1})-D_+^{\gamma}l_1^{\cdot}(d_i)|^{\frac{1}{\xi-\gamma}};\ S_1\geq d_i\right].$$

Furthermore, let (p,q) be a pair of positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Then by applying successively Hölder's inequality and (9), we get

$$d_+'\leq C(H)\sum_j(t_{j+1}-t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}}\sum_{d_i\geq 0}(d_{i+1}-d_i)\mathbb{P}(S_1\geq d_i)^{\frac{1}{q}}$$
$$\leq C(H)\sum_j(t_{j+1}-t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}}\sum_n\sum_{n\leq d_i\leq n+1}(d_{i+1}-d_i)\mathbb{P}(S_1\geq d_i)^{\frac{1}{q}}$$
$$\leq C(H)\sup_j(t_{j+1}-t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}-1}\sum_j(t_{j+1}-t_j)\sum_n\mathbb{P}(S_1\geq n)^{\frac{1}{q}}\ .$$

Since $\frac{1}{2}\leq H<1$, we have $\frac{1-(1+\gamma)H}{\xi-\gamma}>3H>1$, and

$$\sup_j (t_{j+1}-t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}-1} \to 0 \quad \text{as} \quad k \to \infty,$$

then by employing lemma 4.1, we deduce that

$d'_{+} \to 0 \quad as \quad k \to +\infty.$

Also using the same arguments as in d'_{+}, and by symmetry, we obtain

$d'_{-} \to 0 \quad as \quad k \to +\infty.$

Consequently,

$$\mathbb{E}\left[\sum_{i,j} |\Delta_{ij} D^{\gamma}_{+} L|^{\frac{1}{\xi-\gamma}}\right] \underset{k\to\infty}{\longrightarrow} 0. \tag{12}$$

Step2/$\circ$

Now we turn to the case of $D = D^{\gamma}_{-}$.

$$\begin{aligned}
\mathbb{E}\left[\sum_{i,j} |\Delta_{ij} D^{\gamma}_{-} L|^{\frac{1}{\xi-\gamma}}\right] &= \sum_j (t_{j+1}-t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}} \sum_i \mathbb{E}\left[|D^{\gamma}_{-} l^{\cdot}_1(d_{i+1}) - D^{\gamma}_{-} l^{\cdot}_1(d_i)|^{\frac{1}{\xi-\gamma}}\right] \\
&= \sum_j (t_{j+1}-t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}} \sum_{d_i\geq 0} \mathbb{E}\left[|D^{\gamma}_{-} l^{\cdot}_1(d_{i+1}) - D^{\gamma}_{-} l^{\cdot}_1(d_i)|^{\frac{1}{\xi-\gamma}}\right] \\
&+ \sum_j (t_{j+1}-t_j)^{\frac{1-(1+\gamma)H}{\xi-\gamma}} \sum_{d_i\leq 0} \mathbb{E}\left[|D^{\gamma}_{-} l^{\cdot}_1(d_{i+1}) - D^{\gamma}_{-} l^{\cdot}_1(d_i)|^{\frac{1}{\xi-\gamma}}\right] \\
&= d''_{+} + d''_{-}.
\end{aligned}$$

Proceeding exactly as above, we need to prove that $d''_{+} \to 0$and $d''_{-} \to 0$when $k \to \infty$. Let$'$s prove the limit for d''_{-}. By symmetry we may obtain directly the limit for d''_{+}. Note that,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{d_i\leq 0} |D^{\gamma}_{-} l^{\cdot}_1(d_{i+1}) - D^{\gamma}_{-} l^{\cdot}_1(d_i)|^{\frac{1}{\xi-\gamma}}\right] \\
&= \sum_{d_i\leq 0} \mathbb{E}\left[\left|\frac{\gamma}{\Gamma(1-\gamma)}\int_0^{\infty} \frac{l_1^{d_{i+1}-a} - l_1^{d_i-a} - l_1^{d_{i+1}} + l_1^{d_i}}{a^{1+\gamma}} da\right|^{\frac{1}{\xi-\gamma}}, I_1 \leq d_i\right] \\
&+ \sum_{d_i\leq 0} \mathbb{E}\left[\left|\frac{\gamma}{\Gamma(1-\gamma)}\int_0^{\infty} \frac{l_1^{d_{i+1}-a} - l_1^{d_i-a} - l_1^{d_{i+1}} + l_1^{d_i}}{a^{1+\gamma}} da\right|^{\frac{1}{\xi-\gamma}}, I_1, > d_i\right]. \\
&= \mathbb{E}\left[\sum_{d_i\leq 0} |D^{\gamma}_{-} l^{\cdot}_1(d_{i+1}) - D^{\gamma}_{-} l^{\cdot}_1(d_i)|^{\frac{1}{\xi-\gamma}}, I_1 \leq d_i\right],
\end{aligned}$$

where, for the last equality, we have used the fact that, if $I_1 \geq d_i$we have $I_1 \geq d_{i+1} - a$, $I_1 \geq d_i$and $I_1 \geq d_i - a$for any $a \geq 0$.

Then by using same arguments as in the proof of Step1, we get

$$
\begin{aligned}
d''_{-} &= \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}} \mathbb{E}[\sum_{d_i \le 0} |l_1^{d_{i+1}} - l_1^{d_i}|^{\frac{1}{\xi}}, I_1 \le d_i] \\
&\le \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}} \sum_{d_i \le 0} (\mathbb{E}|l_1^{d_{i+1}} - l_1^{d_i}|^{\frac{p}{\xi}})^{\frac{1}{p}} \mathbb{P}[I_1 \le d_i]^{\frac{1}{q}} \\
&\le C(H) \sum_j (t_{j+1}-t_j)^{\frac{1-H}{\xi}} \sum_{d_i \le 0} (d_{i+1}-d_i) \mathbb{P}[I_1 \le d_i]^{\frac{1}{q}} \\
&\le C(H) \sup_j \; (t_{j+1}-t_j)^{\frac{1-H}{\xi}-1} \sum_j (t_{j+1}-t_j) \sum_n \sum_{-n-1 \le d_i \le -n} (d_{i+1}-d_i) \\
&\qquad\qquad \bullet \mathbb{P}[\min_{C_j t_j \le u \le 1 + C_j t_j} B^H(u) \le d_i]^{\frac{1}{q}} \\
&= C(H) \sup_j \; (t_{j+1}-t_j)^{\frac{1-H}{\xi}-1} \sum_j (t_{j+1}-t_j) \sum_n \mathbb{P}[\min_{C_j t_j \le u \le 1 + C_j t_j} B^H(u) \le -n]^{\frac{1}{q}}.
\end{aligned}
$$

In the above inequality we have used the same arguments as in ([11], page 874). So, we have

$$\mathit{sup}_j \; (t_{j+1}-t_j)^{\frac{1-H}{\xi}-1} \to 0 \quad \text{as} \quad k \to \infty,$$

and by lemma 4.1 we can write

$$\sum_n \mathbb{P}[\min_{C_j t_j \le u \le 1 + C_j t_j} B^H(u) \le -n]^{\frac{1}{q}} < +\infty.$$

Hence,

$$d''_{-} \to 0 \quad \text{as} \quad k \to +\infty.$$

Consequently, by symmetry we obtain

$$\mathbb{E}\left[\sum_{i,j} |\Delta_{ij} D_-^\gamma L|^{\frac{1}{\xi-\gamma}} \right] \underset{k\to\infty}{\longrightarrow} 0. \tag{13}$$

Step3/∘

Finally we look at the case of $D = D^\gamma$. Here by the definition, we have

$$D^\gamma = D_+^\gamma - D_-^\gamma,$$

then

$$|D^\gamma| \le |D_+^\gamma| + |D_-^\gamma|.$$

This, jointly considered with (12) and (13), proves the result for $D = D^\gamma$. Here the entire proof completes. ■

Remark 4.5. It is well-known that the Fourier transform $\mathcal{F}$, of fractional derivative of a function $f \in L^1$, is

$$\mathcal{F}(D^\gamma f(x)) = (iw)^\gamma \mathcal{F}f(w).$$

Now, by (2) and the inverse of Fourier transform, we obtain that

$$D^{\gamma}L_t^{\cdot}(x) = \frac{1}{2}\int_{\mathbb{R}}(iu)^{\gamma}\left[\int_0^t e^{iu(x-X_s)}ds\right]du,$$

which neither gives a useful estimation nor the required regularities of fractional derivative of fBm local time, that we need in this paper.

5. Extension to Other Additive Functionals

In this section, we are interested in more general additive functionals (generalized fractional derivative). Let us replace the kernel $\frac{1}{y^{1+\gamma}}$ of $D_{\pm}^{\gamma}$ by a suitable regularly varying function, more precisely, for $0 < \gamma < \xi$,

$$K_{\pm}^{\gamma,l}L_t^{\cdot}(x) = \frac{1}{\Gamma(-\gamma)}\int_0^{+\infty}k_{\gamma,l}(y)\left[L_t^{(x\pm y)} - L_t^x\right]dy, \quad 0 < \gamma < \xi\,, \tag{14}$$

where $k_{\gamma,l} : \mathbb{R} \mapsto [0,+\infty)$ is a regularly varying function of the form

$$k_{\gamma,l} = \begin{cases} l(x)x^{-1-\gamma} & if\ x > 0 \\ 0 & if\ x \leq 0, \end{cases}$$

and l is slowly varying function. (see Fitzsimmons and Getoor [14] and Bingham et al [7]). Then analogous to a fractional derivative by (2), we obtain

$$K_{\pm}^{\gamma,l}L_t^{\cdot}(x) := \frac{1}{\Gamma(-\gamma)}\int_0^{+\infty}\int_{\mathbb{R}}\int_0^t e^{iuX(s)}\left[e^{-iu(x-a)} - e^{-iu(x+a)}\right]\frac{l(a)}{a^{1+\gamma}}dsduda, \tag{15}$$

for $0 < \gamma < \xi$.

Remark 5.1. $K^{\gamma,l}$ has the following scaling property

$$(K^{\gamma,l}L_{\lambda t}^{\cdot}(\lambda^H x),\ t \geq 0,\ x \in \mathbb{R}) \stackrel{\mathfrak{L}}{=} (\lambda^{1-H(1+\gamma)}K^{\gamma,l'}L_t^{\cdot}(x),\ t \geq 0,\ x \in \mathbb{R}),$$

where $l'(.) = l(\lambda^H.)$ a slowly varying function.

In what follows we will give similar results to those of the fractional derivative of fBm local time. For this purpose let's first recall Potter's Theorem, which will play a central role in the proof of our results.

Theorem 5.1. (Potter's Theorem)
i) If l is slowly varying function, then for any chosen constants $A > 1$ and $\xi > 0$, there exists $X = X(A,\xi)$ such that

$$\frac{l(y)}{l(x)} \leq A\max\left\{\left(\frac{y}{x}\right)^{\xi},\left(\frac{y}{x}\right)^{-\xi}\right\}, \quad (x \geq X, \quad y \geq X).$$

ii) If further, l is bounded away from 0 and ∞ on every compact subset of $[0,+\infty)$, then for

every $\xi > 0$*, there exists* $A' = A'(\xi) > 1$ *such that*
$$\frac{l(y)}{l(x)} \leq A' \max\left\{\left(\frac{y}{x}\right)^{\xi}, \left(\frac{y}{x}\right)^{-\xi}\right\}, \quad (x > 0, \quad y > 0).$$

Now, in the following two theorems we deal with the regularities of $K^{\gamma}L$.

Theorem 5.2. *Let* $T > 0$, $0 < \gamma < \xi$ *and let* $K = \{K^{\gamma,l}, K_{\pm}^{\gamma,l}\}$. *Then for all* $m \geq 1$ *there exists a constant* $C > 0$*, such that for every* $0 < t < T$ *and* $(x,y) \in R^2$*, the inequality*
$$\|KL_t^{\cdot}(x) - KL_t^{\cdot}(y)\|_{2m} \leq Ct^{1-H(1+\xi)}|x-y|^{\xi-\gamma}, \tag{16}$$
holds.

Proof. We prove (16) only for the case of $K = K^{\gamma,l}$, since the other cases follow by similar arguments. Let $m \geq 1$, then by (14) and (4), with $s = 0$ and the fact that $L_0^x = 0$, we have
$$\begin{aligned}&\|K^{\gamma,l}L_t^{\cdot}(x) - K^{\gamma,l}L_t^{\cdot}(y)\|_{2m}\\ &\qquad\leq Cl(b)\Big(\int_0^b \frac{l(u)}{l(b)} t^{1-H(1+\xi)} u^{\xi-1-\gamma}\,du + \int_b^{\infty} \frac{l(u)}{l(b)} t^{1-H(1+\xi)}|x-y|^{\xi} u^{-1-\gamma}\,du\Big).\end{aligned}$$

for every $b > 0$.

Moreover, by Potter's Theorem, for all $0 < \delta < max(\xi-\gamma, 2\gamma)$, there exists a positive and finite constant A_δ, such that,
$$\frac{l(u)}{l(b)} \leq A_\delta\left\{\left(\frac{u}{b}\right)^{\delta} \bigvee \left(\frac{u}{b}\right)^{-\delta}\right\}.$$
It follows then that
$$\|K^{\gamma,l}L_t^{\cdot}(x) - K^{\gamma,l}L_t^{\cdot}(y)\|_{2m} \leq Ct^{1-H(1+\xi)}(b^{\xi-\gamma} + |x-y|^{\xi} b^{-\gamma}).$$
Therefore, it suffices now to choose $b = |x-y|$, which gives the desired result. ■

Theorem 5.3. *Under the same assumption of the previous theorem, for all* $m \geq 1$ *there exists a constant* $C > 0$*, such that for every* $0 < t < T$ *and* $(x,y) \in R^2$*, the inequality*
$$\begin{aligned}&\|KL_t^{\cdot}(x) - KL_t^{\cdot}(y) - KL_s^{\cdot}(x) + KL_s^{\cdot}(y)\|_{2m}\\ &\qquad\leq C|t-s|^{1-H(1+\xi)}|x-y|^{\xi-\gamma},\end{aligned} \tag{17}$$
holds.

Proof. For all $m \geq 1$, using (15) we have,
$$\|K^{\gamma}L_t^{\cdot}(x) - K^{\gamma}L_t^{\cdot}(y) - K^{\gamma}L_s^{\cdot}(x) + K^{\gamma}L_s^{\cdot}(y)\|^{2m} \leq C(I_1 + I_2),$$
where

$$I_1 = \|\int_{[0,b]}\int_{\mathbb{R}}\int_{[s,t]} e^{iuB^H(h)}\left[e^{-iu(x-a)} - e^{-iu(x+a)} - e^{-iu(y-a)} + e^{-iu(y+a)}\right]\frac{l(a)}{a^{1+\gamma}} dhduda\|_{2m},$$

and

$$I_2 = \|\int_{[b,+\infty[}\int_{\mathbb{R}}\int_{[s,t]} e^{iuB^H(h)}\left[e^{-iu(x-a)} - e^{-iu(x+a)} - e^{-iu(y-a)} + e^{-iu(y+a)}\right]\frac{l(a)}{a^{1+\gamma}} dhduda\|_{2m},$$

for all $b > 0$.
By using the elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\xi}|\theta|^{\xi}$ for all $0 < \xi < 1$ and any $\theta \in \mathbb{R}$, we obtain

$$I_1^{2m} \leq 2^{-\xi}(l(b))^{2m}\int_{[0,b]^{2m}}\prod_{j=1}^{2m}\frac{l(a_j)}{l(b)}a_j^{\xi-1-\gamma}d\bar{a}$$
$$\cdot\int_{\mathbb{R}^{2m}}\int_{[s,t]^{2m}}\prod_{j=1}^{2m}|u_j|^{\xi}\mathbb{E}\left[e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right]d\bar{h}d\bar{u},$$

and

$$I_2^{2m} \leq |x-y|^{2m\xi}(l(b))^{2m}\int_{[b,+\infty[^{2m}}\prod_{j=1}^{2m}\frac{l(a_j)}{l(b)}a_j^{-1-\gamma}d\bar{a}$$
$$\cdot\int_{\mathbb{R}^{n}}\int_{[s,t]^{2m}}\prod_{j=1}^{2m}|u_j|^{\xi}\mathbb{E}\left[e^{i\sum_{j=1}^{2m}u_jB^H(h_j)}\right]d\bar{h}d\bar{u}.$$

Moreover, by Potter's Theorem, for all $0 < \delta < max(\xi - \gamma, 2\gamma)$, there exists a positive and finite constant A_δ, such that,

$$\frac{l(u)}{l(b)} \leq A_\delta\left\{\left(\frac{u}{b}\right)^{\delta}\bigvee\left(\frac{u}{b}\right)^{-\delta}\right\}.$$

It follows then that

$$I_1^{2m} \leq Cb^{2m(\xi-\gamma)}|t-s|^{2m(1-H(1+\xi))},$$

and

$$I_2^{2m} \leq C|x-y|^{2m\xi}b^{-2m\gamma}|t-s|^{2m(1-H(1+\xi))}.$$

Finally, by choosing $b = |x-y|$, we can write

$$\|K^\gamma L_t^{\cdot}(x) - K^\gamma L_t^{\cdot}(y) - K^\gamma L_s^{\cdot}(x) + K^\gamma L_s^{\cdot}(y)\|_{2m} \leq C\,|t-s|^{1-H(1+\xi)}|x-y|^{\xi-\gamma}.$$

Here the proof completes. ■

In what follows, we give some results about the integration w.r.t. K^γ of local time. Note that by using the same technique for the fractional derivatives of local time of fBm, analogous results for the generalized fractional derivatives of fBm local time are obtained.

Theorem 5.4.
i) For any $\varepsilon > 0$, $K^{\gamma}L_t^{\cdot}(x)$ *is of bounded* $\varepsilon + \frac{1}{\min\left(1, \frac{1}{2H} - \frac{1}{2}\right) - \gamma}$*- variation in x uniformly in t*
ii) For any $\varepsilon > 0$, $K^{\gamma}L_t^{\cdot}(x)$ *is of bounded* $\varepsilon + \frac{1}{1-H(1+\xi)}$*-variation in t uniformly in x.*
iii) For any $p,q > 0$, *such that* $\frac{1}{pq} < \xi - \gamma$ *and* $\frac{1}{p} < 1 - H(1+\xi)$, $K^{\gamma}L$ *is of bounded* p,q*-variations in* (t,x).

Proof. The proof of this theorem is similar to the proof in the fractional derivative case. In particular, we give here the proof of i) in theorem 5.4. So we apply Kolmogorov's criterion to (16), to have almost surely, for every $0 < \delta < \xi - \gamma$, a constant $C > 0$ such that

$$|KL_t^{\cdot}(x) - KL_t^{\cdot}(y)| \leq C\,|x-y|^{\delta},$$

and by the fact that every β –Hölder function is of finite $\frac{1}{\beta}$ – variation on every compact interval, then we conclude that $KL_t^{\cdot}(x)$ is of p –variation in x uniformly in t, for any $p > \frac{1}{\xi-\gamma}$. ■

Remark 5.2. The integral $\int_a^b f(x)dK^{\gamma}L_t^{\cdot}(x)$ is well defined as a pathwise Riemann-Stieltjes integral for any function f of q variation such that $1 \leq q < 1 + \frac{\xi-\gamma}{1+(\varepsilon-1)(\xi-\gamma)}$, for all $\varepsilon > 0$.

Theorem 5.5. *Let* $E = [0,t] \times [c,d]$ *and* $0 < \xi < \min(1, \frac{1}{2H} - \frac{1}{2})$. *Then almost surely the two parameter integral of generalized fractional derivative of fBm local time*

$$\int_0^t \int_c^d f(s,x)dK^{\gamma}L_s^x := \lim_{|E_{n,m}|\to 0} \sum_{i=0}^{n} \sum_{j=0}^{m} f(t_i,x_j)\Delta_{i,j}K^{\gamma}L,$$

is well defined, for any function $f \in C_E^{\alpha_1,\alpha_2}$ *(the space of mixed Hölder function on E), with* $1 \geq \alpha_1 > H(1+\xi)$ *and* $1 \geq \alpha_2 > 1 - \xi + \gamma$. *Moreover, there exists a constant* $C(\alpha_1,\alpha_2,\beta_1,\beta_2) > 0$ *such that the inequality*

$$\left|\int_0^t \int_c^d f(s,x)dK^{\gamma}L_s^x\right| \leq C(\alpha_1,\alpha_2,\beta_1,\beta_2)t^{\beta_1}(d-c)^{\beta_2},$$

with $0 < \beta_1 < 1 - H(1+\xi)$ *and* $0 < \beta_2 < \xi - \gamma$, *is satisfied.*

Remark 5.3. For any function $f \in \mathcal{C}_E^{\alpha_1,\alpha_2}$, with compact support, where $1 \geq \alpha_1 > H(1+\xi)$ and $1 \geq \alpha_2 > 1 - \xi + \gamma$, we can define the integral $J_t := \int_0^t \int_{\mathbb{R}} f(s,x)dK^{\gamma}L_s^{\cdot}(x)$, precisely

$$J_t = \lim_{|En,m|\to 0} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(t_i,x_j)\Delta_{i,j}K^{\gamma}L.$$

Now following the same steps as for fractional derivative of local time, we are able to give in the following theorem the p,q-variation for $K^{\gamma}L$.

Theorem 5.6. With the same assumptions as theorem 4.2, when $K = \{K^{\gamma,l}, K_{-}^{\gamma,l}, K_{+}^{\gamma,l}\}$, the following limit

$$\sum_{(i,j)} |\Delta_{ij} KL|^{\frac{1}{\xi-\gamma}} \underset{k\to\infty}{\overset{L^1}{\longrightarrow}} 0,$$

should hold.

Remark 5.4. These results remain true for the subfractional Brownian motion and bifractional Brownian motion. In general they remain true for all selfsimilar processes which admit local time satisfying the regularities (3) and (4).

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