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Backward Doubly Stochastic Differential Equations With Monotone and Discontinuous Coefficients

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Abstract. *In this paper, we use the Yoshida approximation to prove the existence and uniqueness of a solution for the backward doubly stochastic differential equation when the generator is monotone and continuous. Before that we present the results for existence and uniqueness of an adapted solution of the backward doubly stochastic differential equation under some generals conditions.*

Key words : Backward DSDEs, Monotone Coefficients, Discontinuous Coefficients, Adapted Solution, Existence and Uniqueness.

AMS Subject Classifications : 60H10, 60H05

1. Introduction

A new kind of backward stochastic differential equations was introduced by Pardoux and Peng [13] in 1994, which is a class of backward doubly stochastic differential equation (BDSDEs in short) with two different directions of stochastic integrals, i.e., equations involving both a standard forward stochastic integral and a backward stochastic integral. That is, BDSDEs are stochastic differential equations of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \tag{1}$$

where the dW is a forward Itô integral and the dB is a backward Itô integral. The terminal value ξ and the functions f, g are supposed to be given. In [13], the existence and uniqueness of the solution are established under a uniformly Lipschitz condition on the coefficients. It is worth noting that the definition of solution of this type of equations is slightly different from that of classical BSDEs. The BDSDEs (1) can be related to semilinear and quasilinear stochastic partial differential equations (SPDEs). A link that was developed in many papers (see e. g. [1, 2, 3, 9, 12]) and has motivated many efforts to establish the existence and uniqueness of solutions under more general conditions than the global Lipschitz one as done in [4, 14, 15]. The equation for the adjoint process in optimal stochastic control of [10] is a linear version of the following equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T [h(s, Y_s) + Z_s]dW_s. \tag{2}$$

In this paper, we study the BDSDEs with monotone and continuous coefficients using the Yosida approximation [11] and we obtain the existence and uniqueness of solution for BDSDEs (1). First, we establish the existence and uniqueness for an adapted solution $(Y_t, Z_t)_{0\leq t\leq T}$ of equation (2). As well known, this result should be useful in optimal stochastic control.
We also prove the existence and the uniqueness of the solution for the general equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T h(s, Y_s, Z_s)dW_s, \tag{3}$$

where h is $\mathcal{P} \otimes \mathcal{B}_d \otimes \mathcal{B}_{d\times k}/\mathcal{B}_{d\times k}$ measurable, the mapping $z \longrightarrow h(s,y,z)$ is a bijection for any (s,ω,y).

The paper is organized as follows. In Sections 2, we state some notations, assumptions and define a solution to the BDSDE. In Section 3, we present our first results for existence and uniqueness of an adapted solution under certain conditions. Finally in section 4, we established the existence and uniqueness of the solution in the case where the generator is continuous and monotone.

2. Notation, Preliminaries and Assumptions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $T > 0$. Let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motions defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively. For $t \in [0, T]$, we define
$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \quad and \quad \mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B$,
where $\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$ and $\mathcal{F}_{t,T}^B = \sigma(B_s - B_t; t \leq s \leq T)$, completed with P-null sets. It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub σ –fields, and hence it is not a filtration. However $(\mathcal{G}_t)$ is a filtration. Denote by $\mathcal{P}$ the σ –algebra of $\mathcal{F}_t$ –progressively measurable subsets of $\Omega \times [0, T]$.
Let $M^2(0, T, \mathbb{R}^d)$ denote the set of d –dimensional, jointly measurable stochastic processes $\{\varphi_t; t \in [0, T]\}$, that satisfy :
(a) $E\left(\int_0^T |\varphi_t|^2 dt\right) < \infty$.
(b) φ_t is $\mathcal{F}_t$ –measurable, for any $t \in [0, T]$.

Definition 2.1. The solution of equation (1) is a couple (Y, Z) which belongs in $M^2(0, T, \mathbb{R}^d) \times M^2(0, T, \mathbb{R}^{d\times k})$ and satisfies equation (1).

Next we consider the following assumptions:
H1) $g : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^{d\times l}$ be measurable, $g(.,Y, Z) \in M^2(0, T, \mathbb{R}^{k\times l})$, and there exists a constants $L > 0$ and $0 < \lambda < 1$, such that for every $(\omega, t) \in \Omega \times [0, T]$ and $(Y, Z) \in$

$\mathbb{R}^d \times \mathbb{R}^{d\times k}$,
$\left|g(t,Y,Z) - g\left(t,Y',Z'\right)\right|^2 \leq L\left|Y - Y'\right|^2 + \lambda\left|Z - Z'\right|^2$.
H2) Let ξ be a square integrable random variable which is $\mathcal{F}_T$ − measurable.
H3) Let $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ be $\mathcal{P} \otimes \mathcal{B}_d \otimes \mathcal{B}_{d\times k}/\mathcal{B}_d$ −measurable,
a) $f(.,0,0) \in M^2(0,T,\mathbb{R}^d)$
b) There exist $C > 0$ such that
$|f(t,Z) - f(t,Z')| \leq C|Z - Z'|$.
H4) Let $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ be $\mathcal{P} \otimes \mathcal{B}_d \otimes \mathcal{B}_{d\times k}/\mathcal{B}_d$ −measurable and $h : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ be $\mathcal{P} \otimes \mathcal{B}_d/\mathcal{B}_{d\times k}$ −measurable,
a) $f(.,0,0) \in M^2(0,T,\mathbb{R}^d)$, $h(.,0) \in M^2(0,T,\mathbb{R}^{d\times k})$
b) There exist $C > 0$ such that for all $Y,\ Y' \in \mathbb{R}^d,\ Z,\ Z' \in \mathbb{R}^{d\times k}$,
$|f(t,Y,Z) - f(t,Y',Z')| \leq C(|Y - Y'|+|Z - Z'|)$,
$|h(t,Y) - h(t,Y')| \leq C|Y - Y'|$.
H5) Let $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ be $\mathcal{P} \otimes \mathcal{B}_d \otimes \mathcal{B}_{d\times k}/\mathcal{B}_d$ −measurable and $h : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^{d\times k}$ be $\mathcal{P} \otimes \mathcal{B}_d \otimes \mathcal{B}_{d\times k}/\mathcal{B}_{d\times k}$ −measurable,
a) $f(.,0,0) \in M^2(0,T,\mathbb{R}^d)$, $h(.,0,0) \in M^2(0,T,\mathbb{R}^{d\times k})$,
b)There exist $C > 0$ such that
$|f(t,Y,Z) - f(t,Y',Z')|+|h(t,Y,Z) - h(t,Y',Z''|+|Z - Z'|)$,
for all $Y,\ Y' \in \mathbb{R}^d,\ Z,\ Z' \in \mathbb{R}^{d\times k}$ (t,ω) $a.e.$
c) There exist $\beta > 0$ such that
$|h(t,Y,Z) - h(t,Y,Z''|$,
for all $Y \in \mathbb{R}^d,\ Z,\ Z' \in \mathbb{R}^{d\times k}$ (t,ω) $a.e.$

3. Adapted Solution of Backward Doubly Stochastic DEs

In this section, let us consider the following lemma before the prove of the existence and uniqueness of such solution for equation (2). We start by considering that f, g and h are independent of (Y,Z).

Lemma 3.1. *Under assumption (H2),* $f \in M^2(0,T,R^d)$, $g \in M^2(0,T,R^{k\times l})$,*and* $h \in M^2(0,T,R^{d\times k})$, (Y,Z) *is a solution for the following equation:*

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s)dB_s - \int_t^T [h(s) + Z_s]dW_s. \tag{4}$$

Proof. We define the filtration $\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B$ and the $\mathcal{G}_t$ −square integrable martingale $Y_t = E^{\mathcal{G}_t}[\xi + \int_0^T f(s)ds + \int_0^T g(s)dB_s]$, $0 \leq t \leq T$. The martingale representation theorem yields the existence of $\mathcal{G}_t$ − progressively measurable process $\bar{Z}_t$ with values in $\mathbb{R}^{d\times k}$ such that $E\int_0^T |\bar{Z}_s|^2 ds < \infty$ and

$$Y_t = Y_0 + \int_0^t \bar{Z}_s dW_s.$$

If $Z_t = \bar{Z}_t - h(t)$, $0 \leq t \leq T$, then

$$Y_t = Y_0 + \int_0^t (h(s) + Z_s)dW_s\ ,$$

and
$Y_T = Y_0 + \int_0^T (h(s) + Z_s) dW_s.$
As in the same proof of [13], it follows that $(Y_t, Z_t)_{0\le t\le T}$ is $\mathcal{F}_t$ –adapted and solves equation (4). ■

Now, we consider the following equation:

$$Y_t = \xi + \int_t^T f(s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T [h(s) + Z_s] dW_s. \qquad (5)$$

Proposition 3.1. *Under assumptions (H1), (H2) and (H3), (Y, Z) is the unique solution for equation (5).*

Proof. *(Uniqueness):* Let $\Delta Y_t = Y_t - Y'_t$, and $\Delta Z_t = Z_t - Z'_t$ such that (Y_t, Z_t) and (Y'_t, Z'_t) be two solutions of (5). Using Itô's formula for $|\Delta Y_t|^2$, we get

$$\begin{aligned}|\Delta Y_t|^2 &= 2\int_t^T (\Delta Y_s, f(s, Z_s) - f(s, Z'_s)) ds + 2\int_t^T (\Delta Y_s, g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)) dB_s \\ &\quad - 2\int_t^T (\Delta Y_s, \Delta Z_s) dW_s + \int_t^T |g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)|^2 ds - \int_t^T |\Delta Z_s|^2 ds.\end{aligned}$$

Taking the expectation, leads to

$$\begin{aligned}E|\Delta Y_t|^2 + E\int_t^T |\Delta Z_s|^2 ds &= 2E\int_t^T (\Delta Y_s, f(s, Z_s) - f(s, Z'_s)) ds \\ &\quad + E\int_t^T |g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)|^2 ds.\end{aligned}$$

Then by assumptions (H1) and (H3), we have

$$\begin{aligned}E|\Delta Y_t|^2 + E\int_t^T |\Delta Z_s|^2 ds &\le 2CE\int_t^T |\Delta Y_s||\Delta Z_s| ds + E\int_t^T (L|\Delta Y_s|^2 + \lambda|\Delta Z_s|^2) ds \\ &\le \bar{C}E\int_t^T |\Delta Y_s|^2 ds + \lambda E\int_t^T |\Delta Z_s|^2 + \tfrac{1}{2}E\int_t^T |\Delta Z_s|^2.\end{aligned}$$

Application, here of Gronwall's Lemma, yields the uniqueness of the solution.

(Existence): By the Picard iteration and lemma 3.1, we introduce an approximating sequence as follow: $Z_t^0 \equiv 0$ and $(Y_t^n, Z_t^n)_{n\ge 1}$ be a sequence in $\mathcal{M}^2(t, T; \mathbb{R}^d) \times \mathcal{M}^2(t, T; \mathbb{R}^{d\times k})$ defined by :

$$Y_t^n = \xi + \int_t^T f(s, Z_s^{n-1}) ds + \int_t^T g(s, Y_s^{n-1}, Z_s^{n-1}) dB_s - \int_t^T [h(s) + Z_s^n] dW_s. \qquad (6)$$

Let $\beta \in \mathbb{R}$, with integration by parts applied to $|Y_t^{n+1} - Y_t^n|^2 e^{\beta t}$ we obtain,

$$\begin{aligned}|Y_t^{n+1} - Y_t^n|^2 e^{\beta t} + \beta\int_t^T |Y_s^{n+1} - Y_s^n|^2 e^{\beta s} ds &= 2\int_t^T (Y_s^{n+1} - Y_s^n)(f(s, Z_s^n) - f(s, Z_s^{n-1})) e^{\beta s} ds \\ &\quad + 2\int_t^T (Y_s^{n+1} - Y_s^n)(g(s, Y_s^n, Z_s^n) - g(s, Y_s^{n-1}, Z_s^{n-1})) e^{\beta s} dB_s \\ &\quad - \int_t^T (Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n) e^{\beta s} dW_s\end{aligned}$$

$$+\int_t^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s^{n-1},Z_s^{n-1})|^2 e^{\beta s}ds$$
$$-\int_t^T |Z_s^{n+1}-Z_s^n|^2 e^{\beta s}ds.$$

Taking the expectation, leads to

$$\begin{aligned}E|Y_t^{n+1}-Y_t^n|^2e^{\beta t}+\beta E\int_t^T|Y_s^{n+1}-Y_s^n|^2e^{\beta s}ds+E\int_t^T|Z_s^{n+1}-Z_s^n|^2e^{\beta s}ds\\ = 2E\int_t^T(Y_s^{n+1}-Y_s^n)(f(s,Z_s^n)-f(s,Z_s^{n-1}))e^{\beta s}ds\\ +E\int_t^T|g(s,Y_s^n,Z_s^n)-g(s,Y_s^{n-1},Z_s^{n-1})|^2e^{\beta s}ds.\end{aligned}$$

By assumption (H1) and (H3) we get,

$$\begin{aligned}E|Y_t^{n+1}-Y_t^n|^2e^{\beta t}+\beta E\int_t^T|Y_s^{n+1}-Y_s^n|^2e^{\beta s}ds+E\int_t^T|Z_s^{n+1}-Z_s^n|^2e^{\beta s}ds\\ \le \varepsilon C^2E\int_t^T|Y_s^{n+1}-Y_s^n|e^{\beta s}ds+\frac{1}{\varepsilon}E\int_t^T|Z_s^n-Z_s^{n-1}|e^{\beta s}ds\\ +LE\int_t^T|Y_s^n-Y_s^{n-1}|^2e^{\beta s}ds+\lambda E\int_t^T|Z_s^n-Z_s^{n-1}|^2e^{\beta s}ds.\end{aligned}$$

Now if we choose $\varepsilon=\dfrac{2}{1-\lambda}$ and $\beta=\dfrac{2L}{1+\lambda}+\dfrac{2C^2}{1-\lambda}$, we may deduce that

$$\begin{aligned}E|Y_t^{n+1}-Y_t^n|^2e^{\beta t}+(\frac{2L}{1+\lambda})E\int_t^T|Y_s^{n+1}-Y_s^n|^2e^{\beta s}ds+E\int_t^T|Z_s^{n+1}-Z_s^n|^2e^{\beta s}ds\\ \le \frac{1+\lambda}{2}E\int_t^T(\frac{2L}{1+\lambda}|Y_s^n-Y_s^{n-1}|^2+|Z_s^n-Z_s^{n-1}|^2)e^{\beta s}ds.\end{aligned}$$

It follows then that

$$\begin{aligned}E\int_t^T(\frac{2L}{1+\lambda}|Y_s^{n+1}-Y_s^n|^2+E\int_t^T|Z_s^{n+1}-Z_s^n|^2)e^{\beta s}ds\\ \le (\frac{1+\lambda}{2})^n\,E\int_t^T(\frac{2L}{1+\lambda}|Y_s^1-Y_s^0|^2+|Z_s^1-Z_s^0|^2)e^{\beta s}ds.\end{aligned}$$

Hence (Y^n) and (Z^n) are two Cauchy sequences in $\mathcal{M}^2(0,T,\mathbb{R}^d)\times\mathcal{M}^2(0,T,\mathbb{R}^{d\times k})$. By (6), (Y^n) is also a Cauchy sequence in $\mathcal{L}^2(\Omega,,\mathcal{C}(0,T,\mathbb{R}^d))$. Applying the limit to (6), as $n\to\infty$, we obtain that $Y=\lim_{n\to\infty}Y^n$ and $Z=\lim_{n\to\infty}Z^n$ solve equation (5). ■

In this part of the paper, we shall try to prove the existence and uniqueness of the solution to equation (2).

Theorem 3.1. *Under assumptions (H1), (H2) and (H3), equation* (2) *admits a unique solution.*

Proof. (Uniqueness): Let $\Delta Y_t=Y_t-Y'_t$ and $\Delta Z_t=Z_t-Z'_t$, such that (Y_t,Z_t) and (Y'_t,Z'_t) be two solutions of (2). By applying Itô's formula to $|\Delta Y_t|^2$, we get

$$E|\Delta Y_t|^2+E\int_t^T|\Delta Z_s|^2ds=2E\int_t^T(\Delta Y_s,f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s))ds$$

$$+ E\int_t^T |g(s,Y_s,Z_s) - g(s,Y'_s,Z'_s)|^2 ds$$
$$- 2E\int_t^T (\Delta Z_s, h(s,Y_s) - h(s,Y'_s))ds$$
$$- E\int_t^T |h(s,Y_s) - h(s,Y'_s)|^2 ds.$$

Using assumptions (H1) and (H4), leads to

$$E|\Delta Y_t|^2 + (1-\lambda)E\int_t^T |\Delta Z_s|^2 ds \le 2CE\int_t^T |\Delta Y_s|(|\Delta Y_s|+|\Delta Z_s|)ds$$
$$+ E\int_t^T (L|\Delta Y_s|^2 + \lambda|\Delta Z_s|^2)ds$$
$$- 2CE\int_t^T |\Delta Z_s||\Delta Y_s|ds - CE\int_t^T |\Delta Y_s|^2 ds$$
$$\le \bar{C}E\int_t^T |\Delta Y_s|^2 ds,$$

for a certain $\bar{C}$; and the result follows.

(Existence): Let $Y_t^0 \equiv 0$ and $(Y_t^n, Z_t^n)_{n\ge1}$ be a sequence in $\mathcal{M}^2(t,T,\mathbb{R}^d) \times \mathcal{M}^2(t,T,\mathbb{R}^{d\times k})$ defined by :

$$Y_t^n = \xi + \int_t^T f(s,Y_s^{n-1},Z_s^{n-1})ds + \int_t^T g(s,Y_s^{n-1},Z_s^{n-1})dB_s$$
$$- \int_t^T [h(s,Y_s^{n-1}) + Z_s^{n-1}]dW_s. \tag{7}$$

Application of Itô's formula to $|Y_t^{n+1} - Y_t^n|^2$ and utilization of assumptions (H1) and (H4) lead to

$$E|Y_t^{n+1} - Y_t^n|^2 \le 2CE\int_t^T |Y_s^{n+1} - Y_s^n|(|Y_s^n - Y_s^{n-1}|+|Z_s^n - Z_s^{n-1}|)ds$$
$$+ E\int_t^T (L|Y_s^n - Y_s^{n-1}|^2 + \lambda|Z_s^n - Z_s^{n-1}|^2)ds$$
$$- C^2E\int_t^T |Y_s^n - Y_s^{n-1}|^2 ds - E\int_t^T |Z_s^n - Z_s^{n-1}|^2 ds.$$

Taking $\bar{C}$ to depend only on C and L, leads to

$$E|Y_t^{n+1} - Y_t^n|^2 + (1-\lambda)E\int_t^T |Z_s^n - Z_s^{n-1}|^2 ds \le \bar{C}(E\int_t^T |Y_s^{n+1} - Y_s^n|^2 ds + E\int_t^T |Y_s^n - Y_s^{n-1}|^2 ds).$$

We also have

$$-E|Y_t^{n+1} - Y_t^n|^2 - \bar{C}E\int_t^T |Y_s^{n+1} - Y_s^n|^2 ds \le \bar{C}E\int_t^T |Y_s^n - Y_s^{n-1}|^2 ds.$$

Then

$$E\int_t^T |Y_s^{n+1} - Y_s^n|^2 ds \le \bar{C}\int_t^T \exp(\bar{C}(s-t))\Big[E\int_t^T |Y_s^n - Y_s^{n-1}|^2 ds\Big]ds.$$

The inequality above can be approximated by

$$E\int_0^T |Y_t^{n+1} - Y_t^n|^2 dt \le \frac{\exp(\bar{C}T)}{n!}E\int_0^T |Y_s^n - Y_s^{n-1}|^2 ds.$$

This implies that (Y^n) and (Z^n) are two Cauchy sequences in $\mathcal{M}^2(0,T;\mathbb{R}^d) \times \mathcal{M}^2(0,T;\mathbb{R}^{d\times k})$. Moreover, (7) illustrates that (Y^n) converges in $\mathcal{L}^2(\Omega;\mathcal{C}(0,T;\mathbb{R}^d))$. So we obtain that $Y = \lim_{n\to\infty} Y^n$ and $Z = \lim_{n\to\infty} Z^n$ solve equation (2). ∎

Now, let's find a solution to equation (3).

Theorem 3.2. *Under the assumptions (H1), (H2) and (H5), there exists a unique solution to equation (3).*

Proof. Let us study the equation

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T h(s, Z_s)dW_s, \tag{8}$$

when g satisfies the assumption (H1), $f \in \mathcal{M}^2(0, T, \mathbb{R}^d)$, and $h \in \mathcal{M}^2(0, T, \mathbb{R}^{d\times k})$. The uniqueness and existence of a solution to (8) were studied in lemma 3.1.
Next, we may consider the following equation:
$Y_t = \xi + \int_t^T f(s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T h(s, Z_s)dW_s,$
where f, g, h satisfy the assumptions (H1) and (H3). The uniqueness and the existence of a solution to this equation were proved in proposition 3.1.

Finally, we try to solve equation (3), starting with equation (8). Indeed by lemma 3.1, there exists a unique solution to
$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T \bar{Z}_s dW_s,$
where $\bar{Z} \in \mathcal{M}^2(0, T, \mathbb{R}^{d\times k})$. Hence there exists a unique $Z \in \mathcal{M}^2(0, T, \mathbb{R}^{d\times k})$ such that $h(t, Z_t) = \bar{Z}_t$, (t, ω) $a.e.$, from the assumption for h that for any $(t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^{d\times k}$, there exists a unique element $\phi_t(\omega, z) \in \mathbb{R}^{d\times k}$ such that $h(t, \omega, \phi_t(\omega, z)) = Z$. We show now that ϕ is $\mathcal{P} \otimes \mathcal{B}_{d\times k}/\mathcal{B}_{d\times k}$ –measurable. We assume that $\Omega = ([0, T]; \mathbb{R}^k)$, $W_t(\omega) = \omega_t$, and $\mathcal{F}_T$ is the Borel field over Ω. Then the mapping $H(t, \omega, z) = (t, \omega, h(t, \omega, z))$ is a bijection from $[0, T] \times \Omega \times \mathbb{R}^{d\times k}$ into itself. Since $[0, T] \times \Omega \times \mathbb{R}^{d\times k}$ is a complete and separable metric space, from Theorem 10.5, page 506 in Ethier and Kurtz [8], we deduce that H^{-1} is Borel measurable. For each t the restriction of the same map to $[0, T] \times \mathcal{C}([0, T]; \mathbb{R}^{d\times k}) \times \mathbb{R}^{d\times k}$, we obtain that H^{-1} is $\mathcal{P} \otimes \mathcal{B}_{d\times k}$ measurable and we get that ϕ is $\mathcal{P} \otimes \mathcal{B}_{d\times k}/\mathcal{B}_{d\times k}$ –measurable. ■

4. BDSDEs Under Monotonicity and General Increasing Growth Conditions

Throughout this section we shall consider the following assumptions.
H6) $g : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^{d\times l}$ be measurable, $g(., Y, Z) \in M^2(0, T, \mathbb{R}^{k\times l})$, and there exists a constant $0 < \lambda < 1$, such that for every $(\omega, t) \in \Omega \times [0, T]$ and $(Y, Y', Z, Z') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \times \mathbb{R}^{d\times k}$,
$|g(t, Y, Z) - g(t, Y', Z')|^2 \le \lambda(|Y - Y'|^2 + |Z - Z'|^2)$.
H7) f is continuous in (y, z) for almost all (t, ω), $f(., y.z) \in \mathcal{M}^2(0, T, \mathbb{R}^d)$
a) For all $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d\times k}$, there exists $C_1 > 0$ such that
$|f(t, y, z)| \le |f(t, 0, 0)| + C_1(|Y| + |Z|)$.
b) For all $(y, y', z, z') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \times \mathbb{R}^{d\times k}$, there exists $\frac{7}{4} < C < 2$ such that
$(y - y', f(t, y, z) - f(t, y', z')) \le (1 - C)|z - z'^2 - C|y - y'^2$.

Our main result is reported in this section.

Lemma 4.1. *let* $\Phi : R^n \to R^n$ *be a continuous function and there exists a constant* $c > 0$ *such*

that
$(\Phi(x^1)-\Phi(x^2),\ x^1-x^2)\le -c\ |\ x^1-x^2\ |^2\ ,\quad \forall x^1,x^2\in\mathbb{R}^n.$
Then for the Yosida approximation Φ^α *of* Φ, $\alpha>0$, *we have*
(i) $\left(\Phi^\alpha(x^1)-\Phi^\alpha(x^2),\ x^1-x^2\right)\le -c\ |\ x^1-x^2\ |^2$

$$|\ \Phi^\alpha(x^1)-\Phi^\alpha(x^2)\ |\le(\tfrac{2}{\alpha}+c)\ |\ x^1-x^2\ |$$

$$|\ \Phi^\alpha(x)\ |\le|\ \Phi(x)\ |+2c\ |\ x\ |$$

(ii) *For any* $\alpha,\beta>0$, *we have*

$$(\Phi^\alpha(x^1)-\Phi^\beta(x^2),x^1-x^2)\le(\alpha+\beta)(|\ \Phi(x^1)\ |+|\ \Phi(x^2)\ |+c\ |\ x^1\ |+c\ |\ x^2\ |)^2-c\ |$$

(iii) *For any* $(x^\alpha)_{\alpha>0}\subset R^n$, $x\in R^n$, *if* $\lim\limits_{\alpha\to 0}x^\alpha=x$, *then*

$$\lim_{\alpha\to 0}\Phi^\alpha(x^\alpha)=\Phi(x).$$

This lemma can be proved by adopting the proof given in Da Prato and Zabczyk [5, 6].

Theorem 4.1. *Under assumptions* $(H2),(H6)$ *and* $(H7)$, *there exists a unique adapted solution* $(Y_t,Z_t)\in M^2(0,T,R^d)\times M^2(0,T,\mathbb{R}^{d\times k})$ *for equation* (1).

Proof. (Uniqueness): Let (Y_t,Z_t) and (Y'_t,Z'_t) be two solutions of equation (1). We call $\Delta Y_t=Y_t-Y'_t$ and $\Delta Z_t=Z_t-Z'_t$. Applying the Itô formula to $|\ \Delta Y_t\ |^2$, we get

$$\begin{aligned}E\ |\ \Delta Y_t\ |^2\ &=2E\int_t^T|\ \Delta Y_s\ ||\ f(s,Y_s,Z_s)-f(t,Y'_s,Z'_s\ |\ ds\\&+E\int_t^T|\ g(s,Y_s,Z_s)-g(t,Y'_s,Z'_s)\ |^2\ ds-E\int_t^T|\ \Delta Z_s\ |^2\ ds.\end{aligned}$$

Then by assumptions $(H6)$ and $(H7)$, we have

$$\begin{aligned}E\ |\ \Delta Y_t\ |^2\ -E\int_t^T|\ \Delta Z_s\ |^2\ ds&\le-(2C-\lambda)E\int_t^T|\ \Delta Y_s\ |^2\ ds\\&-(2C-\lambda)E\int_t^T|\ \Delta Z_s\ |^2\ ds.\end{aligned}$$

Hence

$$(2C-\lambda)E\int_t^T[|\ \Delta Y_s\ |^2\ +|\ \Delta Z_s\ |^2\ ]ds-E\int_t^T|\ \Delta Z_s\ |^2\ ds\le 0,$$

for $\frac{7}{4}\le C\le 2$ and $0\le\lambda\le 1$. Then we have $E\int_t^T|\ \Delta Y_s\ |^2\ ds=0$ and $E\int_t^T|\ \Delta Z_s\ |^2\ ds=0$.

Thus there exists a unique adapted solution $(Y_t,Z_t)_{0\le t\le T}\in M^2(0,T,\mathbb{R}^d)\times M^2(0,T,\mathbb{R}^{d\times k})$ for equation (1).

(Existence): Assume $h(t,y,z)\equiv z,$ then equation (1) becomes like (3). To prove the existence, we need to pass through the following four steps.
***Step* 1.** There exists a unique adapted solution for the approximating BDSDEs (3), and for arbitrary $\alpha>0$ the approximating BDSDEs of (3)is

$$\begin{aligned}Y_t^\alpha&=\xi+\int_t^T f^\alpha(s,Y_s^\alpha,Z_s^\alpha)ds+\int_t^T g(s,Y_s^\alpha,Z_s^\alpha)dB_s\\&-\int_t^T h^\alpha(s,Y_s^\alpha,Z_s^\alpha)dW_s,\qquad 0\le t\le T,\end{aligned}\tag{9}$$

where $(f^\alpha(t,Y_t^\alpha,Z_t^\alpha),-h^\alpha(t,Y_t^\alpha,Z_t^\alpha))$ is the Yosida approximation of $(f(t,Y_t,Z_t),-h(t,Y_t,Z_t))$.

Then by lemma 4.1 we get,
$|f^\alpha(t,y,z)-f^\alpha(t,y',z'^2+|h^\alpha(t,y,z)-h^\alpha(t,y',z'^2 \leq (\frac{2}{\alpha}+c)^2(|\ y-y'^2+|z-z'^2).$
Hence
$2|f^\alpha(t,y,z)-f^\alpha(t,y',z'^2+2|h^\alpha(t,y,z)-h^\alpha(t,y',z'^2 \leq 2(\frac{2}{\alpha}+c)^2(|\ y-y'^2+|z-z'^2).$
Furthermore
$(|f^\alpha(t,y,z)-f^\alpha(t,y',z'^\alpha(t,y,z)-h^\alpha(t,y',z'^2 \leq 2(\frac{2}{\alpha}+c)^2(|\ y-y'^2+|z-z'^2).$
Therefore

$$\begin{aligned} &|\ f^\alpha(t,y,z)-f^\alpha(t,y',z'^\alpha(t,y,z)-h^\alpha(t,y',z')\ | \\ &\qquad \leq \sqrt{2}\,(\tfrac{2}{\alpha}+c)(|\ y-y'^2+|\ z-z'^2)^{\frac{1}{2}} \\ &\qquad \leq \sqrt{2}\,(\tfrac{2}{\alpha}+c)(|\ y-y'\ |+|\ z-z'\ |), \end{aligned} \tag{10}$$

and by lemma 4.1, we have
$(f^\alpha(t,y,z)-f^\alpha(t,y',z'^\alpha(t,y,z'^\alpha(t,y,z),z-z') \leq -c\,\big|\,z-z'\big|^2,$
which implies that
$|\ h^\alpha(t,y,z)-h^\alpha(t,y,z')\ ||\ z-z'\ | \geq c\,\big|\,z-z'\big|^2.$
Then

$$|\ h^\alpha(t,y,z)-h^\alpha(t,y,z')\ | \geq c\ |\ z-z'\ |. \tag{11}$$

Clearly (10) and (11) yield the assumptions of theorem 3.2. Hence, there exists a unique adapted solution (Y^α, Z^α) for equation (9) in $M^2(0,T,\mathbb{R}^d)\times M^2(0,T,\mathbb{R}^{d\times k})$.
***Step* 2.** Estimation of $Y^\alpha,\ Z^\alpha$. By applying the Itô formula to $|\ Y_t^\alpha\ |^2$, we obtain

$$\begin{aligned} E\xi^2 = E\ |\ Y_0^\alpha\ |^2\ -2E\int_0^T |\ Y_s^\alpha\ ||\ f^\alpha(s,Y_s^\alpha,Z_s^\alpha)\ |\ ds - E\int_0^T |\ g(s,Y_s^\alpha,Z_s^\alpha)\ |^2\ ds \\ + E\int_0^T |\ h^\alpha(s,Y_s^\alpha,Z_s^\alpha)\ |^2\ ds. \end{aligned}$$

So

$$\begin{aligned} E\ |\ Y_0^\alpha\ |^2\ +2CE\int_0^T(|\ Y_s^\alpha\ |^2\ +|\ Z_s^\alpha\ |^2\ )ds \leq E\xi^2 + E\int_0^T |\ Y_s^\alpha\ |^2\ ds + E\int_0^T |Z^\alpha|^2 ds \\ + E\int_0^T |\ g(s,Y_s^\alpha,Z_s^\alpha)\ |^2\ ds + E\int_0^T |\ f^\alpha(s,0,0)\ | \\ -2E\int_0^T (Z_s^\alpha, h^\alpha(s,0,0))ds \\ +2E\int_0^T |h^\alpha(s,0,0)|ds + \tfrac{1}{2}E\int_0^T |Z_s^\alpha|^2 ds. \end{aligned}$$

On another hand, we have
$|\ f^\alpha(t,Y_t^\alpha,Z_t^\alpha)\ | \leq |f(t,Y_t^\alpha,Z_t^\alpha)|+2C(|Y_t^\alpha|+|Z_t^\alpha|).$
Then
$|\ f^\alpha(t,0,0)\ |^2\ +|\ h^\alpha(t,0,0)\ |^2\ \leq |\ f(t,0,0)|^2.$
Therefore

$$\begin{aligned} &E|Y_0^\alpha|^2 + (2C-\tfrac{3}{2}-\lambda)E\int_0^T(|Y_s^\alpha|^2+|Z_s^\alpha|^2)ds \\ &\qquad \leq E\xi^2 + E\int_0^T |\ g(s,0,0)\ |^2\ ds + 2E\int_0^T |\ f^\alpha(s,0,0)\ |^2\ ds. \end{aligned}$$

Ultimately for $\frac{7}{4} \leq C \leq 2$, if we let

$$K = \left[2E\xi^2 + 2E\int_0^T |\ g(s,0,0)\ |^2\ ds + 4E\int_0^T |\ f^\alpha(s,0,0)\ |^2\ ds\right]/(4C-3-2\lambda),$$

then

$E\int_t^T(\mid Y_s^\alpha \mid^2 + \mid Z_s^\alpha \mid^2)ds \le K.$

***Step* 3.** (Y^α, Z^α) converge in $M^2(0,T,\mathbb{R}^d)\times M^2(0,T,\mathbb{R}^{d\times k})$. Consider $\alpha>0$ and $\beta>0$ then apply Itô's formula to $\mid Y_t^\alpha - Y_t^\beta \mid^2$ to obtain

$$E \mid Y_0^\alpha - Y_{0t}^\beta \mid^2 + 2E\int_0^T (Y_s^\alpha - Y_s^\beta, f^\beta(s,Y_s^\beta,Z_s^\beta) - f^\alpha(s,Y_s^\alpha,Z_s^\alpha))ds$$
$$-E\int_0^T \mid g(s,Y_s^\alpha,Z_s^\alpha) - g(s,Y_s^\beta,Z_s^\beta) \mid^2 ds + E\int_0^T \mid h^\alpha(s,Y_s^\alpha,Z_s^\alpha) - h^\beta(s,Y_s^\beta,Z_s^\beta) \mid^2 ds = 0.$$

Hence

$$\begin{aligned}
E \mid Y_0^\alpha - Y_0^\beta \mid^2 &- 2E\int_0^T (Z_s^\alpha - Z_s^\beta, h^\alpha(s,Y_s^\alpha,Z_s^\alpha) - h^\beta(s,Y_s^\beta,Z_s^\beta))ds\\
&+ E\int_0^T \mid h^\alpha(s,Y_s^\alpha,Z_s^\alpha) - h^\beta(s,Y_s^\beta,Z_s^\beta) \mid^2 ds\\
&\le -2CE\int_0^T (|Y_s^\alpha - Y_s^\beta|^2 + |Z_s^\alpha - Zs^\beta|^2)ds\\
&+ \lambda E\int_0^T (|Y_s^\alpha - Y_s^\beta|^2 + |Z_s^\alpha - Zs^\beta|^2)ds\\
&+ 2(\alpha+\beta)E\int_0^T \Big[(|f^\alpha(s,Y_s^\alpha,Z_s^\alpha)| + |f^\beta(s,Y_s^\beta,Z_s^\beta)|) + |h^\alpha(s,Y_s^\alpha,Z_s^\alpha)|\\
&+ |h^\beta(s,Y_s^\beta,Z_s^\beta)| + C(|Y_s^\alpha| + |Z_s^\alpha| + |Y_s^\beta| + |Z_s^\beta|) \Big]^2 ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
E \mid Y_0^\alpha - Y_0^\beta \mid^2 + (2C-\lambda)E\int_0^T (|Y_s^\alpha - Y_s^\beta|^2 + |Z_s^\alpha - Z_s^\beta|^2)ds\\
\le 2E\int_0^T (Z_s^\alpha - Z_s^\beta, h^\alpha(s,Y_s^\alpha,Z_s^\alpha) - h^\beta(s,Y_s^\beta,Z_s^\beta))ds\\
- E\int_0^T \mid h^\alpha(s,Y_s^\alpha,Z_s^\alpha) - h^\beta(s,Y_s^\beta,Z_s^\beta) \mid^2 ds\\
+ 2(\alpha+\beta)E\int_0^T \Big[|f^\alpha(s,Y_s^\alpha,Z_s^\alpha)| + |f^\beta(s,Y_s^\beta,Z_s^\beta)|) + |h^\alpha(s,Y_s^\alpha,Z_s^\alpha)|\\
+ |h^\beta(s,Y_s^\beta,Z_s^\beta)| + C(|Y_s^\alpha| + |Z_s^\alpha| + |Y_s^\beta| + |Z_s^\beta| \Big]^2 ds\\
- E\int_0^T |Z_s^\alpha - Z_s^\beta|^2 ds + E\int_0^T |Z_s^\alpha - Z_s^\beta|^2 ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
E \mid Y_0^\alpha - Y_0^\beta \mid^2 + (2C-\lambda-1)E\int_0^T (|Y_s^\alpha - Y_s^\beta|^2 + |Z_s^\alpha - Zs^\beta|^2)ds\\
\le 8(\alpha+\beta)E\int_0^T \Big[(|f^\alpha(s,Y_s^\alpha,Z_s^\alpha)|^2 + |f^\beta(s,Y_s^\beta,Z_s^\beta)|^2) + |h^\alpha(s,Y_s^\alpha,Z_s^\alpha)|^2\\
+ |h^\beta(s,Y_s^\beta,Z_s^\beta)|^2 + C^2(|Y_s^\alpha|^2 + |Z_s^\alpha|^2 + |Y_s^\beta|^2 + |Z_s^\beta|^2) \Big] ds.
\end{aligned}$$

Using $\mid f(t,y,z) \mid \le \mid f(t,0) \mid + C_1(\mid y \mid + |z|)$ and $E\int_0^T(\mid y \mid^2 + |z|^2)dt \le K$, we deduce that there exists a constant $K' > 0$, such that

$$E \mid Y_0^\alpha - Y_0^\beta \mid^2 + (2C-\lambda-1)E\int_0^T (|Y_s^\alpha - Y_s^\beta|^2 + |Z_s^\alpha - Zs^\beta|^2)ds \le K'(\alpha+\beta).$$

If $\frac{7}{4} \le C \le 2$, then $(Y^\alpha, Z^\alpha)_{\alpha>0}$ is a Cauchy sequence in $M^2(0,T,\mathbb{R}^d)\times M^2(0,T,\mathbb{R}^{d\times k})$; and we denote its limit by $(Y,Z) \in M^2(0,T,\mathbb{R}^d)\times M^2(0,T,\mathbb{R}^{d\times k})$.

***Step* 4.** Involves taking weak limits in the approximating equation (9). From lemma 4.1 and assumptions $(H6)$, $(H7)a)$, there exist constant l and m such that

$$| f^\alpha(t,Y_t^\alpha,Z_t^\alpha)+g(t,Y_t^\alpha,Z_t^\alpha)+h^\alpha(t,Y_t^\alpha,Z_t^\alpha) |^2 \leq l(| f(t,0,0) |^2 +| g(t,0,0) |^2)+m(| Y_t^\alpha |^2 +| Z_t^\alpha |^2).$$

So, there exists a constant $C_2 > 0$ such that

$$E\int_t^T | f^\alpha(s,Y_s^\alpha,Z_s^\alpha)+g(s,Y_s^\alpha,Z_s^\alpha)+h^\alpha(s,Y_s^\alpha,Z_s^\alpha) |^2 ds \leq C_2.$$

Therefore there exists a subsequence of

$(f^\alpha(.,y^\alpha,z^\alpha),g(.,y^\alpha,z^\alpha),h^\alpha(.,y^\alpha,z^\alpha))_{\alpha>0}$

which converges weakly to $(F,\bar{g},H)$ in the space $M^2(0,T,\mathbb{R}^d\times\mathbb{R}^{d\times l}\times\mathbb{R}^{d\times k})$. The weak limits in the approximating equation (9) yield

$$Y_t = \xi+\int_t^T F_s ds+\int_t^T \bar{g}(s)dB_s-\int_t^T H_s dW_s.$$

Similar to the proof by Darling and Pardoux [7], it remains to prove that

$F_t = f(t,Y_t,Z_t),\ \bar{g}(t) = g(t,Y_t,Z_t),\ H_t = h(t,Y_t,Z_t),\ P-a.s.,\ a.e.\ t\in[0,T].$ Therefore

$$Y_t = \xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T g(s,Y_s,Z_s)dB_s-\int_t^T Z_s dW_s.$$

Then $(Y_t,Z_t)_{0\leq t\leq T}$ is an adapted solution of equation (1). Here the existence proof ends. ■

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